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The Rotation Age, the Constrained Faustmann Problem and the Initial Conditions

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The traditional unrestricted forest rotation problem as presented by JOHANSSON and LÖFGREN [4] is generalized[1]) and compared with different kinds of restricted harvest problems. The restrictions are of two kinds: 1. Harvest capacity and 2. A constant harvest level.

It is proved that (how) the optimal harvest level and the optimal selection of stands are dependent on the initial age distribution, regeneration costs, price function, the rate of interest and the harvest capacity input requirements. Analytical and numerical methods are used to derive the optimal solutions and the comparative statics results. Some stand selection rules for simple restricted situations are derived.

1. Introduction

The purpose of this paper is to highlight some fundamental properties of the optimal solutions to different kinds of restricted harvest problems in forestry. The paper deals with deterministic problems only. This is an obvious drawback as stressed by for instance LOHMANDER, [8]. However, the restricted deterministic problems have gained some attention lately (compare HULTKRANTZ [2] and JACOBSSON [3]). In this paper, we will find that their results are consistent with very special planning situations only. The intention is to generalize the models and to pinpoint the critical assumptions to the different results. Most likely, it will in the future be possible to extend the analysis of this paper to stochastic *and* restricted cases.

The restricted rotation problems have been discussed earlier by for instance von MALMBORG [13]. He explicitly took the capacity restrictions into account via linear programming. KILKKI [5] also uses linear programming in the planning of forest harvesting. LOHMANDER [6] presents the principles of budgeting within the forest industry enterprise and the use of linear programming. The technical problem solving including the "*standard*" restrictions are given. LOHMANDER [7] analyses the harvest level restriction problem in forestry and shows that it is economical to deviate from the distribution suggested by HULTKRANTZ [2].

The analysis of the "*traditional*" and unrestricted rotation problem is well presented by CLARK [1] and by JOHANSSON and LÖFGREN [4]. This paper also includes a generalization[1]) of their analysis. The reason partly is that this serves as a valuable reference solution to the restricted problems.

[1]) Age dependent net prices are introduced.

2. A Selection of Restricted Situations

Let us select a limited set of problems. From a scientific point of view it is not very satisfactory to do so. One general model would be the most interesting problem. However in order to generalize the existing and very different models and in order to limit the analysis to some partial but interesting cases, some different models are used. However, in all of them, the present value is maximized, which is motivated on the grounds suggested by Johansson and Löfgren [4] in Chapter 1.

The first problem is (1), which is a generalized unrestricted rotation problem (compare [4]). The generalization simply is that the price is a general function of *age* here. This is obviously an important model change since it explicitly captures an important part of forest economics, namely price "growth". In the model used by Johansson and Löfgren, price was treated as a constant (which was age independent). A *time* trend was however discussed[2]).

$$\pi = [-C + e^{-rT} P(T) V(T)] \frac{1}{1 - e^{-rT}}. \tag{1}$$

π is the present value of an infinite series of identical forest generations, t denotes time, C is the regeneration cost, $P(T)$ is the net price, r is the rate of interest, T is the rotation age and $V(T)$ is the stand density; see Fig. 1.

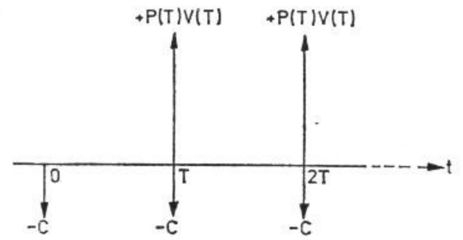

Fig. 1. The unrestricted problem according to Equ. (1)

The next problem (2) is the following: You own a forest area without any forest. You want to maximize the present value of all future profits and have the restriction that when harvesting begins, you can never more change the harvest level. What is the optimal rotation period under these conditions? It should be clear that the problem definition suffers from the following;

- There may be an initial forest on the land. The model cannot deal with the optimal harvest of that initial forest. Furthermore, the initial forest may severely influence the optimal harvest period in the future. This will be shown in this paper.
- Generally, harvest level restrictions have to do with the coordination with existing capacity in forest industry or harvest resources. In this model, the restrictions have nothing to do with the present flow. The harvest is restricted only after one rotation period and in the future. It is unlikely that the mill capacity or the harvest machine capacity can be perfectly predicted maybe 100 years in advance.

2) The distinction is important since an age dependence may exist even if no time trend can be detected in the market price.

Anyhow, this model is motivated here since it is a strong generalization of the HULTKRANTZ [2] model and it shows the sensitivity of that model results to the unrealistic assumptions. In the model analysed by Hultkrantz, the price was constant and no costs were present.

$$\pi = \frac{1}{T}\left\{\int_0^T - \mathrm{e}^{-rt}C\,\mathrm{d}t + \int_0^T \mathrm{e}^{-rt}\,[P(T)V(T) - C]\frac{1}{\mathrm{e}^{rT}-1}\,\mathrm{d}t\right\} \tag{2}$$

Because of the particular underlying assumptions, it is suggested that the model (2) is called the "*Iceland forestry problem*".
($1/T$ on the RHS in (2) is the part of the forest which is harvested and replanted each year if the rotation period is T years; see Fig. 2.)

A third problem is defined in (3). This problem corresponds to problem (2) with respect to the future conditions. However, the initial forest is explicitly taken into consideration.

$$\pi = I(T; x, T_0) + \mathrm{e}^{-rx}F(T). \tag{3}$$

F in Equ. (3) is defined as π in Equ. (2). $I(T; x, T_0)$ is the present value of the profits from the initial forest, T is the rotation age in the future forest (according to Equ. (2)), x is the point in time when the regeneration of the new forest starts and T_0 is a parameter describing the density in the initial forest; see Fig. 3.
(A special case is of course if the initial forest land is evenly distributed over the forest age spectrum and T_0 denotes the age of the oldest trees.)

Note that (3) is a much more general formulation than (2). The basic improvement of realism is that the initial forest is taken into consideration. The sooner the old forest is harvested, the sooner the new forest can be established. As will be shown, this relationship generally influences the optimal rotation age in the new forest.

In (4) the problem is the following; – You should buy a forest area with an initial forest. The area is evenly distributed over the ages. What is the optimal initial age of the oldest stand? (The harvest level must be constant in the future.) Note that (4) is quite different from (2) in the sence that you can select the rotation age in both cases

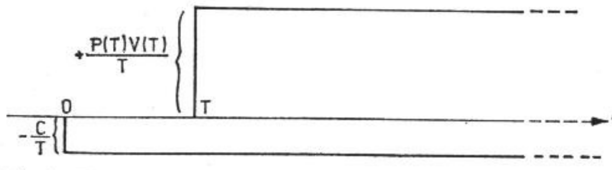

Fig. 2. The restricted problem according to Equ. (2)

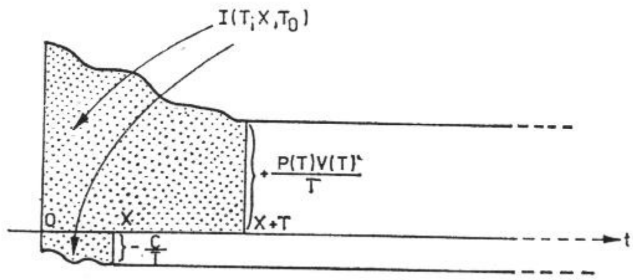

Fig. 3. The restricted problem according to Equ. (3)

but if you select a high rotation age in (2), then you have to wait long for the harvest to begin. In case (4) you can start harvesting at once. This of course affects the optimal rotation ages, which are different in the two "*similar*" problems.

$$\Pi = \frac{1}{T}\int_0^T e^{-rt}\,[P(T)V(T) - C]\,\frac{1}{1 - e^{-rT}}\,dt. \tag{4}$$

Finally we analyse the more traditional kind of restriction, namely restricted harvest capacity.

$$\Pi = f_{11}h_{11} + f_{12}h_{12} + \dots + f_{1n}h_{1n} + f_{21}h_{21} + f_{22}h_{22} + \dots + f_{2n}h_{2n}. \tag{5}$$

Equ. (5) shows the present value of the profits from harvest in n different stands. f_{ij} denotes the present value per hectare of harvesting stand j in period i. h_{ij} denotes the number of hectares in stand j that are harvested in period i. The problem contains only two periods, where we assume that the harvest capacity is restricted in period 1 only, according to (6). We assume (which is quite likely in a real world situation) that the harvest capacity can be adjusted in the future.

$$\begin{aligned}
&\alpha_{11}h_{11} + \alpha_{12}h_{12} + \dots + \alpha_{1n}h_{1n} \le A_1,\\
&\alpha_{21}h_{11} + \alpha_{22}h_{12} + \dots + \alpha_{2n}h_{1n} \le A_2,\\
&\quad\vdots\\
&\alpha_{m1}h_{11} + \alpha_{m2}h_{12} + \dots + \alpha_{mn}h_{1n} \le A_m.
\end{aligned} \tag{6}$$

A particular area is harvested in period 1 or period 2. The total area in one stand is denoted by H_j. f_{1j} is strictly greater than f_{2j}, which means that all stands under investigation should be harvested in period 1 if there would be sufficient harvest capacity. Period 1 is so short that no stand can be completely harvested during that period. ("*Stand*" may be thought of as "*stand class*" or "*forest*".)

Even though this problem may seem highly restricted, it is a quite general problem which deals with the typical situation faced by the forest owner;

- What stands should I harvest now, considering my present restrictions in machine capacity and labour time?

It will turn out that simple and quite general rules of optimal stand selection can be found. Furthermore, the optimal selection of stands cannot be made without knowledge of the present value if harvest takes place today and if harvest takes place in the unrestricted period, the input coefficients and the different capacities. Jacobsson [3] selects stands without regarding the resource requirements, which obviously leads to a solution which is not optimal when there are capacity restrictions. Furthermore, if there are no restrictions, there is no reason to select stands. All stands could be harvested at once!

3. Results

The optimization problems that result from the Equs. (1), (2), (3), (4) and (5 and 6) are solved in the corresponding appendices. The rule which gives the optimal rotation age and the most interesting comparative statics results are derived.

In Appendix 6, the simple optimization program used in the construction of the Figs. 4, ..., 13 is given. In the graphs (and the program), there are two different optimization problems under investigation. The different selections are SEL 1 and SEL 2,

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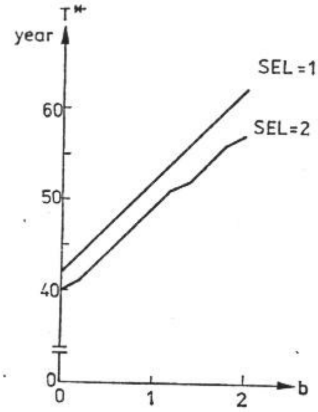

Fig. 4. The optimal rotation period as a function of the derivative of net price with respect to age.
$P(t) = a + bt$, $P(50) = 100$ crowns/m³

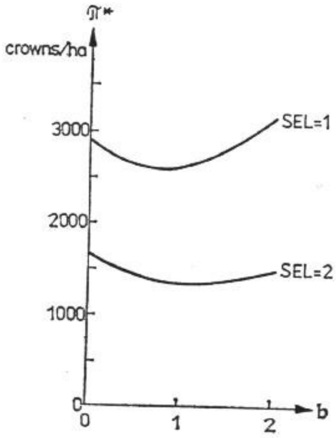

Fig. 5. The optimal present value as a function of the derivative of net price with respect to age.
$P(t) = a + bt$, $P(50) = 100$ crowns/m³

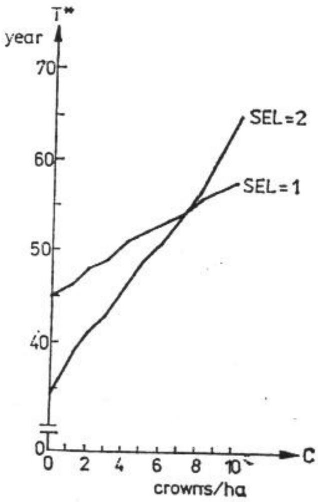

Fig. 6. The optimal rotation age as a function of the cost of regeneration

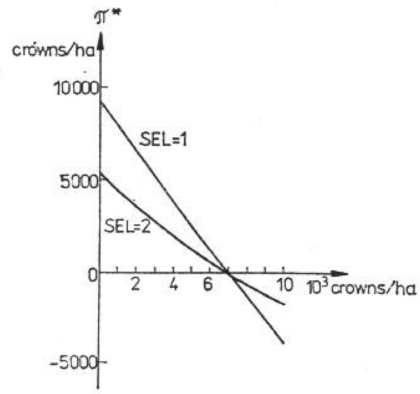

Fig. 7. The optimal present value as a function of the cost of regeneration

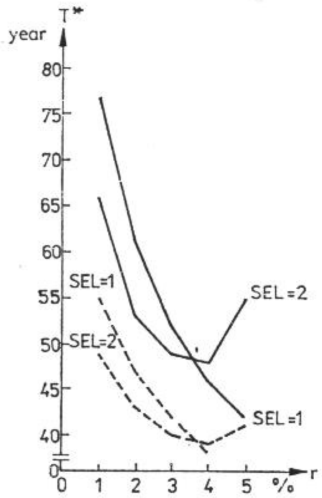

Fig. 8. The optimal rotation period as a function of the rate of interest.

$P(t) = 50 + t$(solid line)

$P(t) = 100$ (dotted line)

which correspond to the Equs. (1) and (2) respectively. The volume function (stand density as a function of age) is discussed in more detail by LOHMANDER [8]. It is shown in (7).

In the numerical examples (the Figs. 4, ..., 13), the parameters are given the following numerical values if nothing else is written in the figure texts:
$C = 5\,000$ crowns/hectare, $P(T) = (50 + T)$ crowns per cubic metre, $r = 3\%$, $M = 6.4$ m³/hectare and $S = 60$ years. (The parameters $M = 6.4$ and $S = 60$ correspond to Pinus Contorta, $H50 = 20$ m.)

$$V(t) = M * S * (1.6416)\,(1. - 6.3582^{(-t/S)})^{2.8967}, \tag{7}$$

where $V(t)$ is stand density at age t, M is the maximum mean annual increment and S is the rotation age of maximum mean annual increment.

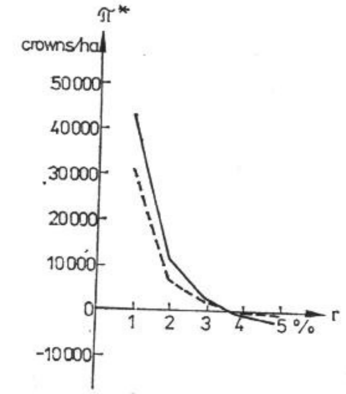

Fig. 9. The optimal present value as a function of the rate of interest (—: SEL = 1; – –: SEL = 2)

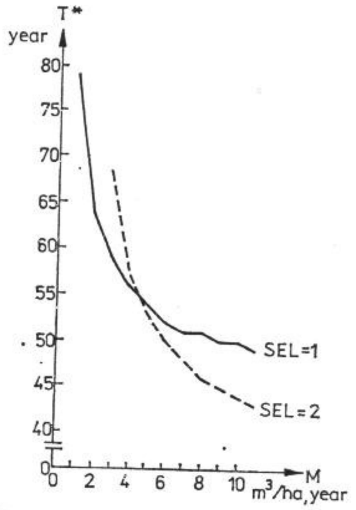

Fig. 10. The optimal rotation age as a function of M, the maximum mean annual increment

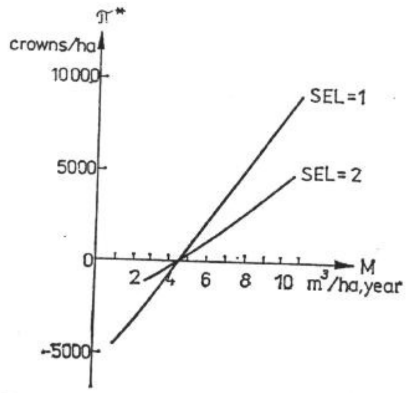

Fig. 11. The optimal present value as a function of M, the maximum mean annual increment

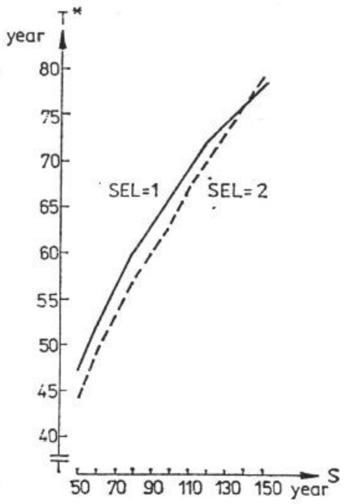

Fig. 12. The optimal rotation age as a function of S, the rotation age which gives maximum mean annual increment

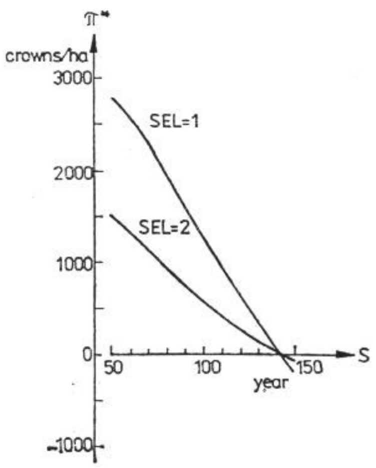

Fig. 13. The optimal present value as a function of S, the rotation age which gives maximum mean annual increment

3.1. Results from Equation (1)

In Appendix 1, we find that the optimal rule is the following

$$\frac{P'(T)V(T) + P(T)V'(T)}{\{-C + P(T)V(T)\}\dfrac{1}{1 - e^{-rT}}} = r. \tag{8}$$

In the rest of this section, T denotes the optimal rotation age. The interpretation of (8) is that the value growth in the forest (at T) should give exactly the rate of interest to the value of the existing stand (which is released during the harvest) and the present value of all future forest generations.

The difference between (8) and the corresponding solution in JOHANSSON and LÖFGREN [4] is simply the presence of $P'(T)V(T)$ in the nominator. Hence, in this version, the price growth[3]) gives some extra value growth in this case and the optimal rotation age should generally be higher. In Appendix 1, it is shown that, ceteres paribus, then the optimal rotation age also increases. The corresponding quantitative results are presented in Figs. 4 and 5.

The optimal rotation age increases with the regeneration cost and decreases with the rate of interest in the capital market. These effects are the same in the model used by Johansson and Löfgren. The numerical results are shown in the Figs. 6 and 8.

Particularily, one should observe the following:

- The optimal rotation age is very sensitive to the slope of the net price function. The optimal present value, on the other hand, is rather insensitive to the slope. (When the price slope is very high, the optimal rotation age is even higher than the age which gives the maximum mean annual increment.)
- The optimal rotation age and the optimal present value are very sensitive to the regeneration cost. In the particular example, 7000 crowns per hectare is the regeneration cost which gives zero profitability.
- The optimal rotation age and the optimal present value are extremely sensitive to the rate of interest. The rate of interest which gives zero profitability is 3.7% in the example (the internal rate of interest).
- When the maximum mean annual increment increases, then the optimal rotation age decreases and the optimal present value increases. The maximum mean annual increment must be at least 4.5 m^3 hectare if forestry should be profitable in the example.
- When the age of maximum mean annual increment increases, then the optimal rotation age increases and the optimal present value decrease. It is interesting to note that the optimal rotation age increases with about two years when the age of maximum annual increment increases with ten years. If forestry should be profitable, then the age of maximum mean annual increment must be below 142 years.

3.2. Results from Equation (2)

In Appendix 2, we find that the optimal rule is

$$\frac{-Cre^{rT} + P'(T)V(T) + P(T)V'(T)}{-Ce^{rT} + P(T)V(T)} = \frac{1}{T} + r. \tag{9}$$

As a special case of (9), if we exclude costs and age dependent net prices from the calculations and just look at volumes, then we get a result consistent with HULTKRANTZ [2], namely

$$\frac{V'(T)}{V(T)} = \frac{1}{T} + r. \tag{10}$$

However, since we deal with economics, we should not exclude the properties of prices and costs, particularily if they will influence the analytical and numerical results in

[3]) Note particularily that this result has nothing to do with a changing price level on the roundwood market. The optimal rotation age is discussed conditional on the observed price level and the age dependence in the net price function.

such a way as what is shown below. See also Section 3.3., where the analysis is further generalized.

The analytical comparative statics results are the following:

- The optimal rotation age increases with the regeneration cost, decreases with the rate of interest and increases with the slope of the net price function (ceteres paribus). The following results should be particularily observed:
- The optimal rotation age is highly sensitive to the slope of the net price function, but the optimal present value is rather insensitive to this slope.
- The optimal rotation age is more sensitive to the regeneration cost in this case than according to the "*traditional*" (unrestricted) Equ. (1). This means (compare Fig. 6) that the optimal rotation age is shorter, equal to or higher than the rotation age in the corresponding unrestricted problem (Equ. (1)). However, the optimal rotation age is obviously the same in the two models when the optimal present value is equal to zero. (Compare the Figs. 6, 7, 8 and 9.)
- The critical levels of C, r, M and S when profitability is zero are the same for the unrestricted and the restricted models.
- The qualitative results with respect to the parameters C, r, M and S are the same for the unrestricted and the restricted models except for that the optimal rotation age increases with the rate of interest in the restricted model if the optimal present value is negative. The reason (which can be obtained through inspection of the equations in Appendix 2) is that if profitability is negative, then we can decrease the loss by selecting a very high rotation age (decreasing the part of the land which is planted each year). If the rotation age is infinity, then the present value of the costs is zero.
- If the optimal present value is positive, then the unrestricted model gives a higher present value. The reason is that, in the unrestricted model, all of the area can be replanted at once. Hence, the profits will be gained earlier. Furthermore, there is no harvest level restriction, which by itself may lead to higher profitability.

3.3. Results from Equation (3)

In Appendix 3, we find that the optimal rotation age in the future forest (the restricted forest according to Eq. (2)) increases strictly with the density of the initial forest. Hence, if there is an initial forest, the optimal rotation age in the new, restricted forest can not be calculated without regarding the initial conditions. Obviously, if we do so anyway, we can expect a systematic deviation from the true (optimal) rotation age.

3.4. Results from Equation (4)

This model is very similar to the Equ. (2) model. However, in this model, we will start harvesting at once, irrespective of the rotation age. We simply assume that the forest already has the optimal age distribution. Hence, we can expect that the optimal rotation age in this case is higher than in the case resulting from Equ. (2). According to Appendix 4., the optimal rotation age is determined from the condition

$$\frac{P'(T)V(T)+P(T)V'(T)}{P(T)V(T)-C}=\frac{1}{T}. \tag{11}$$

Clearly, if we assume price to be age independent and the regeneration cost to be zero, then (11) is consistent with

$$\frac{V'(T)}{V(T)}=\frac{1}{T}. \tag{12}$$

If (12) is rewritten as (13), we find that the optimal rotation age is the age which maximizes the mean annual increment. Clearly, the rotation age according to (12) is higher than the rotation age suggested by the simplified Equ. (10). This confirmes intuition

$$V'(T) = \frac{V(T)}{T}. \tag{13}$$

It is interesting to observe that the optimal rotation age in this (11) problem is independent of the rate of interest in the capital market! This has to do with the harvest level restriction and the initial condition assumed, that the optimal age distribution already exists; Irrespective of the rate of interest, we want to maximize the profit which we get each year in the future. This yearly profit is dependent on the age of the oldest trees, the reforestration costs and the age dependent price only.

The rotation age (13) is also the optimal solution to the unrestricted rotation problem (1) if the price is age independent, the reforestration cost is zero and the rate of interest approaches zero. Compare JOHANSSON and LÖFGREN [4].

3.5. Results from Equations (5) and (6)

In Appendix 5, two different situations are described. In situation 1, there is only one efficient restriction (one resource that is completely used) in the harvest problem. It is found that only one stand (forest area) should be harvested in that case and that the stand(forest area) should be selected which maximizes the ratio (14). This means that the limited resource is used the most profitable way.

$$\frac{\left\{\begin{array}{l}\text{Present value per hectare}\\ \text{if harvest takes place in}\\ \text{the restricted period}\end{array}\right\} - \left\{\begin{array}{l}\text{Present value per hectare}\\ \text{if harvest takes place in}\\ \text{the unrestricted period}\end{array}\right\}}{\begin{array}{l}\text{Requirement of the limited}\\ \text{resource per hectare in the}\\ \text{restricted period}\end{array}} \tag{14}$$

(14) implies that, in situation 1, it is not possible to select harvest objects optimally without investigating the resource requirements and the different present values of all stands (forest areas). On the other hand, Equ. (14) is quite general and convenient to handle in most cases that belong to situation 1.

In situation 2, there are two limited resources that restrict the optimal solution. The optimal selection of stands can no longer be made in the simple way (as in situation 1). The graphical method suggested in the appendix can easily be used.

It should be clear that the optimal selection of stands is dependent not only on the quantities discussed in situation 1, but also on the available harvest resource quantities. More details are found in Appendix 5. In situation 2, it may be optimal to harvest one or two stands (forest areas). Assume that it is optimal to harvest two stands (forest areas). If the capacity of harvest resource 1 increases, the stand which is the most "*capacity 1 intensive*" should be more intensively harvested. The other stand should be *less* intensively harvested in period 1.

This result is included to show that the optimal selection of stands and the optimal harvest levels in different stands, are dependent on the total optimization problem in quite a complicated way even in "*simple*" situations with a small number of restrictions. In more complicated (and realistic) situations, the optimal selection of stands should be made within a total optimization model explicitly including all restrictions and activities of the firm. Compare LOHMANDER [6].

4. Discussion

Optimal rotation under restrictions is a nice area in the sence that any rotation age can be shown to be optimal in some planning situation! This has been shown in the analysis of this paper. The "*standard*" parameters such as the price function parameters, the regeneration cost and the rate of interest make the solutions even more interesting.

However, this also motivates a warning. In the past, many authors have suggested "*optimal*" harvest ages and rules of thumb. These ages and rules generally show that the typical restrictions and parameters discussed in this paper have not been taken into consideration. Hence, they generally lead to solutions that do not maximize the present value of the firm. This is turn reduces the intertemporal consumption possibilities.

It is recommended that, if restrictions are present, the total optimization problem of the firm is solved in a way which explicitly takes all activities and restrictions into consideration. That way the problem of suboptimization is avoided. The author is convinced that it is better to have one rough optimization model that captures all activities of the firm than to have several partial optimization models with many details. Generally, the total coordination of activities is more important to profitability than the detailed, but maybe wrong, stand selections (that may be the results of partial forest models).

The reader should be aware that the principles of optimal harvesting are quite different in a stochastic environment (future growth and prices can generally not be perfectly predicted). Some recent results in that area can be found in LOHMANDER ([9, 10, 12]). (Optimal intertemporal harvesting under the influence of markets where prices are not completely exogenous is the area of study in LOHMANDER [11]. It is important that future research efforts are directed towards the problem: – How should you manage natural resources in the presence of stochastic parameters *and* temporal restrictions? This is also the area of present investigation by the author of this paper.

Appendix 1

In this appendix Equ. (1) is analysed. A unique maximum with respect to T is assumed.

$$\max_T \pi = [-C + e^{-rT} P(T) V(T)] \frac{1}{1 - e^{-rT}}$$

$$\max_T \pi = -C + \frac{[-C + P(T) V(T)]}{[e^{rT} - 1]}$$

$$\frac{\partial \pi}{\partial T} = \frac{\{P'(T) V(T) + P(T) V'(T)\} \{1 - e^{-rT}\} - \{-C + P(T) V(T)\} r}{(e^{rT} - 1)^2 e^{-rT}}$$

$$= \frac{\overset{=0}{N}}{\underset{>0}{D}} = 0$$

$$(N = 0) \Rightarrow \frac{P'(T) V(T) + P(T) V'(T)}{\{-C + P(T) V(T)\} \frac{1}{1 - e^{-rT}}} = r$$

$$\left(\left(\frac{\partial \Pi}{\partial T}\right) = \frac{N}{D},\ N = 0,\ D > 0\right) \Rightarrow \left(\operatorname{sgn}\left(\frac{\partial^2 \Pi}{\partial T\, \partial x}\right) = \operatorname{sgn}\left(\frac{\partial N}{\partial x}\right)\right)$$

$$\left[\frac{\partial N}{\partial C} = r > 0\right] \Rightarrow \left[\frac{\partial T^*}{\partial C} > 0\right]$$

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Results in Ap

Append

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Definition. $P(t) = a + bt$

$$N = \{bV(T) + (a + bT)V'(T)\}\{1 - e^{-rT}\} - \{-C + (a + bT)V(T)\}r = 0$$

Let $[P(T^*) = a + bT^* = \text{constant}, b \text{ varies}] \Rightarrow [a = P(T^*) - bT^*]$

$$N = \{bV(T) + (P - bT + bT)V'(T)\}[1 - e^{-rT}] - \{-C + [P - bT + bT]V(T)\}r = 0$$

$$\frac{\partial N}{\partial b}\Big|_{P(T^*)\text{fix}} = [V(T)[1 - e^{-rT}] > 0] \Rightarrow \left[\frac{\partial T^*}{\partial b} > 0\right]$$

$$\frac{\partial N}{\partial r} = Te^{-rT}\{bV(T) + (a + bT)V'(T)\} + C - (a + bT)V(T)$$

$$r\frac{\partial N}{\partial r} = \{bV(T) + (a + bT)V'(T)\}\,Tre^{-rT} - \{-C + (a + bT)V(T)\}r$$

$$\left\{r\frac{\partial N}{\partial r} - \underset{=0}{N}\right\} = [Tre^{-rT} - 1 + e^{-rT}]\{bV(T) + (a + bT)V'(T)\}$$

$$= e^{-rT}\{Tr - e^{rT} + 1\}\ \{>0\}$$

$$= e^{-rT}\{1 + Tr - e^{Tr}\}\ \{>0\}$$

$$= \underset{>0}{e^{-x}}\{\underset{<0}{(1 + x) - e^{x}}\}\quad \{>0\} < 0,\ (x = Tr > 0)$$

$$\Rightarrow$$

$$\left[\frac{\partial N}{\partial r} < 0\right] \Rightarrow \frac{\partial T^*}{\partial r} < 0$$

Results in Appendix 1:

$$\frac{P'(T)V(T) + P(T)V'(T)}{\{-C + P(T)V(T)\}\dfrac{1}{1 - e^{-rT}}} = r,\quad \frac{\partial T^*}{\partial C} > 0,\quad \frac{\partial T^*}{\partial r} < 0,\quad \frac{\partial T^*}{\partial b}\Big|_{P(T^*)\text{fix}} > 0$$

Appendix 2

In this appendix Equ. (2) is analysed. A unique maximum with respect to T is assumed.

$$\max_T \Pi = \frac{1}{T}\left\{\int_0^T - e^{-rt}C\,dt + \int_0^T e^{-rt}[P(T)V(T) - C]\frac{1}{e^{rT} - 1}\,dt\right\}$$

$$\max_T \Pi = \frac{1}{T}\left\{-C + [P(T)V(T) - C]\frac{1}{e^{rT} - 1}\right\}\left\{-\frac{e^{-rt}}{r}\Big]_0^T\right\}$$

$$\max_T \Pi = \frac{1}{T}\left\{\frac{-C[e^{rT} - 1] + P(T)V(T) - C}{e^{rT} - 1}\right\}\left\{\frac{1 - e^{-rT}}{r}\right\}$$

$$\max_T \Pi = \frac{-Ce^{rT} + P(T)V(T)}{Tr\,e^{rT}}$$

$$\frac{\partial \Pi}{\partial T} = \frac{\left\{\begin{array}{l}[-rCe^{rT} + P'(T)V(T) + P(T)V'(T)]Tre^{rT} + \\ -[-Ce^{rT} + P(T)V(T)]\,[re^{rT} + Tr^2e^{rT}]\end{array}\right\}}{(Tre^{rT})^2} = 0$$

$$\frac{\partial \Pi}{\partial T} = \frac{T\{-Cre^{rT} + P'(T)V(T) + P(T)V'(T)\} - [1 + Tr]\{-Ce^{rT} + P(T)V(T)\}}{(Tre^{rT})^2\,(re^{rT})^{-1}}$$

$$= \frac{\overset{=0}{N}}{\underset{>0}{D}} = 0$$

$(N = 0) \Rightarrow$

$$\frac{-Cre^{rT} + P'(T)V(T) + P(T)V'(T)}{-Ce^{rT} + P(T)V(T)} = \frac{1}{T} + r$$

$$\left(\frac{\partial \Pi}{\partial T} = \frac{N}{D},\ N = 0,\ D > 0\right) \Rightarrow \left(\operatorname{sgn}\left(\frac{\partial^2 \Pi}{\partial T\,\partial X}\right) = \operatorname{sgn}\left(\frac{\partial N}{\partial X}\right)\right)$$

$$\left[\frac{\partial N}{\partial C} = e^{rT} > 0\right] \Rightarrow \left[\frac{\partial T^*}{\partial C} > 0\right]$$

$$\frac{\partial N}{\partial r} = T[Ce^{rT} - P(T)V(T)] < 0$$

assumed to be <0

$$\left(\frac{\partial N}{\partial r} < 0\right) \Rightarrow \left[\frac{\partial T^*}{\partial r} < 0\right]$$

Definition. $P(t) = a + bt$

$$N = TaV'(T) + T^2bV'(T) + Ce^{rT} - aV(T) - TraV(T) - T^2rbV(T) = 0$$

Let $[P(T^*) = a + bT^* = \text{constant},\ b \text{ varies}] \Rightarrow [a = P(T^*) - bT^*]$

$$N = T(P - bT)V'(T) + T^2bV'(T) + Ce^{rT} - (P - bT)V(T) - Tr(P - bT)V(T)$$
$$- T^2rbV(T) = 0$$

$$\left[\frac{\partial N}{\partial b} = TV(T) > 0\right] \Rightarrow \left[\frac{\partial T^*}{\partial b} > 0\right]$$

Results in Appendix 2:

$$\frac{-Cre^{rT} + P'(T)V(T) + P(T)V'(T)}{-Ce^{rT} + P(T)V(T)} = \frac{1}{T} + r$$

$$\frac{\partial T^*}{\partial C} > 0,\quad \frac{\partial T^*}{\partial r} < 0,\quad \frac{\partial T^*}{\partial b\,|_{P(T^*)\mathrm{fix}}} > 0$$

Appendix 3

In this appendix, Equ. (3) is analysed. A unique maximum with respect to T is assumed.

$$\max_T \Pi = I(T; x, T_0) + e^{-rx}F(T)$$

$$\frac{\partial \Pi}{\partial T} = \frac{\partial I}{\partial T} + e^{-rx}\frac{\partial F(T)}{\partial T} = 0$$

Differentiation of the first order condition with respect to T and T_0 gives:

$$\underbrace{\frac{\partial^2 \Pi}{\partial T^2}}_{X} \mathrm{d}T^* + \underbrace{\frac{\partial^2 \Pi}{\partial T\, \partial T_0}}_{Y} \mathrm{d}T_0 = 0.$$

It is reasonable to assume that Y is strictly positive. As the density of the initial forest increases, it becomes more valuable to increase the number of years during which harvest of that forest should take place, particularily if we have limited harvest or mill capacity (and a negative slope on the demand curve)

$$\frac{\partial T^*}{\partial T_0} = -\frac{\overset{>0}{Y}}{\underset{<0}{X}} > 0.$$

Hence, the optimal rotation age in the future forest is strictly positively affected by the density in the initial forest.

Appendix 4

In this appendix Equ. (4) is analysed. A unique maximum with respect to T is assumed.

$$\max_T \Pi = \frac{1}{T} \int_0^T \mathrm{e}^{-rt}[P(T)V(T) - C] \frac{1}{1 - \mathrm{e}^{-rT}} \mathrm{d}t$$

$$\max_T \Pi = \frac{P(T)V(T) - C}{T(1 - \mathrm{e}^{-rT})} \left(- \frac{\mathrm{e}^{-rt}}{r} \Big]_0^T \right)$$

$$\max_T \Pi = \frac{P(T)V(T) - C}{T(1 - \mathrm{e}^{-rT})} \left\{ \frac{1 - \mathrm{e}^{-rT}}{r} \right\}$$

$$\max_T \Pi = \frac{P(T)V(T) - C}{Tr}$$

$$\frac{\partial \Pi}{\partial T} = \frac{\{P'(T)V(T) + P(T)V'(T)\}Tr - \{P(T)V(T) - C\}r}{(Tr)^2} = 0$$

$$\frac{\partial \Pi}{\partial T} = \frac{\{P'(T)V(T) + P(T)V'(T)\}T - \{P(T)V(T) - C\}}{T^2 r} = \frac{\overset{=0}{N}}{\underset{>0}{D}} = 0$$

$(N = 0) \Rightarrow$

$$\frac{P'(T)V(T) + P(T)V'(T)}{P(T)V(T) - C} = \frac{1}{T}.$$

A special case is when $P'(T) = 0$, $C = 0$.

Then $N = 0 \Rightarrow$

$$\frac{V'(T)}{V(T)} = \frac{1}{T}$$

$$V'(T) = \frac{V(T)}{T} \qquad \text{(The optimal rotation age is equal to the age of MSY.)}$$

$$\left[\frac{\partial N}{\partial C} = 1 > 0\right] \Rightarrow \frac{\partial T^*}{\partial C} > 0$$

$$\left[\frac{\partial N}{\partial r} = 0\right] \quad \Rightarrow \frac{\partial T^*}{\partial r} = 0$$

$$\left[\frac{\partial N}{\partial P'(T)\,|_{P(T^*)\mathrm{fix}}} > 0\right] \Rightarrow \frac{\partial T^*}{\partial P'(T)\,|_{P(T^*)\mathrm{fix}}} > 0$$

Appendix 5

In this appendix Equs. (5) and (6) are analysed.

$$\begin{aligned} \max \Pi = {} & f_{11}h_{11} + f_{12}h_{12} + \dots + f_{1n}h_{1n} + f_{21}h_{21} + f_{22}h_{22} + \dots + f_{2n}h_{2n} \\ \text{s.t.} \quad & \alpha_{11}h_{11} + \alpha_{12}h_{12} + \dots + \alpha_{1n}h_{1n} \le A_1 \\ & \alpha_{21}h_{11} + \alpha_{22}h_{12} + \dots + \alpha_{2n}h_{1n} \le A_2 \\ & \vdots \\ & \alpha_{m1}h_{11} + \alpha_{m2}h_{12} + \dots + \alpha_{mn}h_{1n} \le A_m \end{aligned}$$

Making use of the assumptions in the text, we get:

$$\max \Pi = (f_{11} - f_{21})h_{11} + (f_{12} - f_{22})h_{12} + \dots + (f_{1n} - f_{2n})h_{1n} + \sum_j f_{2j}H_j$$

s.t. {Equ. (6)}.

Situation 1. There is only one harvest resource with restricted capacity. The dual problem is:

$$\begin{aligned} \min C = {} & A_1 s_1 \\ \text{s.t.} \quad & a_{11}s_1 \ge (f_{11} - f_{21}) && \text{(stand 1)} \\ & a_{12}s_1 \ge (f_{12} - f_{22}) && \text{(stand 2)} \\ & \vdots \\ & \alpha_{1n}s_1 \ge (f_{1n} - f_{2n}) && \text{(stand } n\text{)}. \end{aligned}$$

The restrictions can be rewritten as:

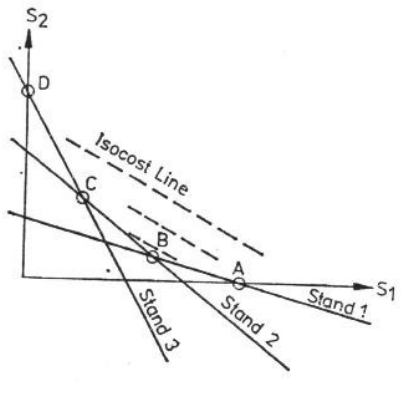
$$s_1 \ge \frac{(f_{1j} - f_{2j})}{\alpha_{1j}}.$$

Hence, if α_{1j}, f_{1j} and f_{2j} are parameters that can take any real value, only one stand will restrict the dual goal function with probability one. The stand which should be harvested in period 1 is the stand which maximizes the ratio $R(j)$

$$R(j) = \frac{(f_{1j} - f_{2j})}{\alpha_{1j}}.$$

Situation 2. There are two harvest resources with restricted capacity. The dual problem is:

$$\begin{aligned} \min C = {} & A_1 s_1 + A_2 s_2 \\ \text{s.t.} \quad & \alpha_{11}s_1 + \alpha_{21}s_2 \ge (f_{11} - f_{21}) && \text{(stand 1)} \\ & \alpha_{12}s_1 + \alpha_{22}s_2 \ge (f_{12} - f_{22}) && \text{(stand 2)} \\ & \vdots \\ & \alpha_{1n}s_1 + \alpha_{2n}s_2 \ge (f_{1n} - f_{2n}) && \text{(stand } n\text{)} \end{aligned}$$



The optimal solution (the optimal selection of stands) depends on the slope of the isocost line (which is a function of the capacities A_1 and A_2) and the coefficients α_{ij} for all i and j. In the graph, the optimal solution is intersection B, which means that stand 1 and stand 2 should be harvested to some extent, since they maximize the object function of the primal problem (and minimize the object function of the dual problem).

It should be clear that if the problem definition is correct, then any method of stand selection which does not take the parameters A_i and α_{ij} into account leads to a solution which is generally not optimal.

Now, assume that B is the optimal intersection (stand 1 and 2 should be harvested). Then, the optimal levels of h_{11} and h_{12} are calculated from:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} h_{11}^* \\ h_{12}^* \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Differentiation with respect to h_{1j}^* and A_i gives;

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} dh_{11}^* \\ dh_{12}^* \end{bmatrix} = \begin{bmatrix} dA_1 \\ dA_2 \end{bmatrix}$$

$$\frac{\partial h_{11}^*}{\partial A_1} = \frac{\begin{vmatrix} 1 & \alpha_{12} \\ 0 & \alpha_{22} \end{vmatrix}}{\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}} = \frac{\alpha_{22}}{(\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})}$$

$$\frac{\partial h_{11}^*}{\partial A_1} = \frac{1}{\alpha_{21}\left(\frac{\alpha_{11}}{\alpha_{21}} - \frac{\alpha_{12}}{\alpha_{22}}\right)}$$

$$\Rightarrow$$

$$\left(\frac{\alpha_{11}}{\alpha_{21}} \gtreqless \frac{\alpha_{12}}{\alpha_{22}}\right) \Rightarrow \left\{\frac{\partial h_{11}^*}{\partial A_1} \gtreqless 0\right\}$$

Consider the following problem. Stand 1 and 2 should be harvested. The capacity of harvest resource 1 increases. The stand which is the most "*capacity 1 intensive*" should then be more intensively harvested. The other stand should be less intensively harvested.

Appendix 6

This computer code was used in the construction of the Figs. 12 and 13. Very similar codes were used in the construction of the Figs. 4, ..., 11.

```
  1 FOR SEL = 1 TO 2
  5 PRINT" CONSTRAINED ROTATION"
  6 PRINT" SELECT ="; SEL
  7 A = 50
  8 B = 1
 10 C = 5000
 20 R = .03
 30 P = 0
 40 M = 6.4
 50 S = 60
 55 FOR X = 1 TO 11
 56 S = 40 + X * 10
 59 LPRINT" SEL, C, P, R, M, S, A, B ="; SEL; C; P; R; M; S; A; B
 60 W = -10000
 70 FOR T = 10 TO 200
 80 V = M * S * 1.6416 * (1 - 6.3582^(-T/S))^2.8967
 85 P = A + B * T
 90 E1 = (-C + EXP(-R * T) * P * V)/(1 - EXP(-R * T))
 91 E2 = (-C * EXP(R * T) + P * V)/(T * R * EXP(R * T))
 95 E = E1
 96 IF SEL = 2 THEN E = E2
100 IF E > W THEN 150
110 U = T - 1
120 PRINT" OPTT ="; U
130 PRINT" PVAL ="; W
135 LPRINT" OPTT ="; U; "P VAL ="; W
140 GOTO 180
150 W = E
160 NEXT T
170 PRINT" MAX T"
180 NEXT X
185 NEXT SEL
190 END
```

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