SAMS, 1997, Vol. 29, pp. 27–55
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Printed in India

THE CONSTRAINED PROBABILITY ORBIT OF MIXED STRATEGY GAMES WITH MARGINAL ADJUSTMENT: GENERAL THEORY AND TIMBER MARKET APPLICATION

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(Received 19 June 1996)

The mixed strategy *n*-person (in particular two person) games have been given much attention in the literature. When all participants have complete information concerning the pay off matrix and all strategies are well and unambiguously defined, the reported standard solutions are relevant and each participant may calculate the optimal probabilities of the different decisions. In most real world situations, however, the elements of the pay off matrix can not be assumed known. In this analysis, each actor optimizes the pure or mixed strategy continuously and marginally, depending on the latest observed decision frequencies of the other actors. A general two person non-zero sum game (with zero sum as a special case) is investigated in this way. Under typical assumptions in "games of conflict", the dynamic solution turns out to be a "constrained cyclical orbit". Hence, it is argued, it is likely that the decision probabilities of the actors in real world games in Nature and in the economy change periodically. A duopsony application where two sawmills are competing in the timber market is included and the dynamics is investigated.

Keywords: Mixed strategy; matrix; probability

1. INTRODUCTION

How will participants, players, in a "game" change behaviour over time in situations when they do not know how the different participants are affected by different possible actions taken by the other players. This general question will be addressed in this paper. Typical applications may be found in economics and ecology. Particularly where there is a small number of participants, or classes of "typical participants" that act in approximately the same way within the class, the theory should be useful.

Some typical market situations to be handled within this framework in economics are oligopolies and oligopsonies, in particular duopolies and duopsonies.

2. GAME THEORY, A FRACTION OF THE LITERATURE AND SOME REFLEXIONS

The classical book on game theory is Luce and Raiffa (1957). This is extremely well written and contains everything from utility theory and many important game problems to mathematics and numerical methods. Of course, Luce and Raiffa did not open a completely new field. Large parts of the theory can also be found in von Neumann and Morgenstern (1944).

Some, but far from all, of the earlier papers of special relevance to the issue treated in this paper are the following:

Cournot (1838) presents a revolutionary contribution to the theory of non cooperative equilibria in oligopoly situations. The ideas of Cournot will partly be used later on in this paper. Luce and Raiffa [22], however, do not mention Cournot at all. Rasmusen [34] pg. 76–78 has rediscovered him. Flåm and Moxnes [14] interpreted the central idea of Cournot as the basis of a dynamic game and developed this theory. Compare also Cavazutti and Flåm [8], Flåm [13] and Flåm and Zaccour [15].

Von Stackelberg [35] and [36] is one of the persons who has contributed to game theory before the concept was established. In particular, he was interested in dynamic duopoly theory. A nice simultaneous treatment of the theories of Cournot and von Stackelberg is found in Henderson and Quandt [17] in Chapter 8.

Nash [24] and [26] gave us the important concept "Nash equilibrium". In a Nash equilibrium, no player has incentive to deviate from his strategy as long as the other players do not deviate from their strategies. The Nash equilibrium has been very useful in most developments of game theory. Some recent refinements of the definitions can be found in Myerson [23].

Brown and von Neumann [5] discussed how to use differential equations in the solution of games. Robinson [31] used an iteration method where each player sequentially estimated the probability distributions of the other players decisions and adapted the own decision probabilities optimally. Brown [4] investigated a problem similar to the problem in Robinson [31]. Bellman [2] continued the studies of iterative algorithms and so did von Neumann [33].

Luce and Raiffa [22] and the earlier authors, mainly treated games that could be classified as static and stationary in the following sense: The economic consequences of different decision combinations were the same over time. The position in time and/or some physical state space did usually not matter.

Dresher [12] stressed the time dimension and optimal decisions over time in connection to several games of conflict.

Isaacs [20] took the step towards a complete description of the time and physical state space. He introduced the theory and several applications of differential games and the sometimes more numerically practical approximation method "difference games". Suddenly it was possible to determine the optimal behaviour over time and space in typical dog fight games of the air force and similar conflicts. The true dynamics had entered the game scene. The three dimensional graphs of the optimal paths of the air planes were tremendous.

However, some problems remained also when the differential games had appeared. First of all, which has become obvious in many applied mathematical problems, we have to restrict the state space in order to be able to handle the numerical calculations in finite time. In difference games, the number of possible positions increases very rapidly with the state space resolution and the size of the area to be represented.

In many cases, it is necessary to calculate the optimal behaviour of each player for each possible position in the physical state space and speed vector and for each possible positions(s), speed vector(s) and decisions(s) of the other players(s). The problem is then solved recursively in the spirit of dynamic programming for every player conditional on the behaviour of all other players. In fact, in a two person difference game, if the decisions of player B or their probability distribution are known by player A and the decisions made by player B are not affected by the decisions made by player A, then player A may regard his optimization problem in the difference game as a common dynamic programming problem. This, however, is a very special case where we do not really investigate games anymore. We then have a "game against Nature". The dimensionality problem in dynamic programming is well known. In difference games, the dimensionality problem is much worse!

Then, what can be done?

If we accept low resolution in the state and time space and a low number of possible decisions (controls), then the difference games can often be rapidly solved. Furthermore, we usually have to assume that the game is deterministic: Each player selects a pure position dependent strategy. If we let the players use randomized strategies, make different decisions with different probabilities in different situations, then the computation time grows very rapidly.

One observation concerning the deterministic differential or difference game is that the outcome is known when the initial conditions are known.

One may argue that solutions of differential equations may be chaotic and that the future state is not predictable. This discussion will not be continued here. The interested reader is suggested to read Prigogine and Stengers [28], Gleick [16] and Puu [29]. In order to solve chaotic differential game systems, it seems necessary to estimate probability density functions of the chaotic outcomes and then treat the problem as a stochastic differential game.

In a deterministic differential or difference game, each player knows exactly what to do and what the other players will do in every possible situation. There is really no need to play the game. For this reason, we may say that we know the outcome of a game of chess as soon as we know what person starts the game with white color.

Of course, in reality, the players do not know much enough or have time enough to calculate the optimal decisions in all possible positions. This is true in chess and in most real world games of some interest. In particular, we should note that the rules of the physical movements (the speed and the possible directions) of the horse or the queen in chess are known exactly by the two players. In real world conflicts, the technical properties of the equipment and the exact positions of the army units may not be known by the opponent. In other kinds of conflicts in a complicated society, the options available to the opponent are frequently very difficult to estimate.

Some highly simplified early differential equation models of conflict with historical comments can also be found in Braun [3] on page 396–411.

Game theory is relevant where there is life and where decisions are made by several organisms with different objectives that influence the same system. Thus, game theory has invaded economics, ecology and the military.

Graduate courses in economics often include some game theory. Chiang (1974) includes a chapter on standard two person zero sum games which unfortunately for some reason has been excluded from later editions. Rasmusen (1990) presents a wide spectrum of game models from economics and related fields but avoids the level of technical detail found in many other publications. The game models found in economics applications are frequently found also in ecology. However, in recent years a large fraction of the ecological game theory has been devoted to the concept of evolutionarily stable strategies. Will a strategy "survive" in the long run in competition with other strategies? Several interesting results have been reported in this area. See in particular Axelrod and Hamilton, Hofstadter [19], Hines [18] and Boyd and Lorberbaum [6].

The following observations can be made:

–In many real world games in economics, ecology and warfare, the pay off consequences (in economic problems such as profit or in ecological problems such as probability of reproduction) of a particular decision combination are not independent of the players positions in time and space. We should in these cases prefer to use game models where time and space are represented if this does not cost too much in the unit of calculation effort. As mentioned, however, such models often become very problematic from many points of view.

–In many real world games, we can not expect that the players know very much about the other players decision options or how the other players valuate different decision combinations.

–In many real world cases, the physical and economic environment of the game problems change rapidly and often unpredictably. One player may own a factory which produces a particular product. If the price of the product is high, this player may be very interested to buy a unit of some input factor. The input factor transaction may be a game in which the factory owner participates among other potential buyers. In this case, the factory owner highly valuates a decision combination which means that he may buy the input factor. One month later, the price of the product decreases dramatically. Again, the factory owner participates in a similar transaction game. This time he does not valuate a decision combination which makes him buy the input factor very highly anymore.

Since the economic environment unpredictably changes in this game, we can not expect that the players will select the same strategy for ever. Hence, we can not be sure that a player who estimates the probabilities of other players decisions via the frequencies in the complete historical decision observations series, and optimizes his strategy accordingly, will optimize his expected result in the changing environment. Robinson [31] utilizes the complete historical decision observation series.

We should in many cases prefer game models that are based on the following assumptions:

- Each player knows how he, or his enterprise, is affected by each decision combination.
- The players do not known how the other players are affected by different decision combinations.
- The players sequentially observe the decisions made by the other players.
- Each player estimates the probabilities that the other players make different decisions using only the latest N observations. N is a number which is small enough to make sure that only observations from the latest and relevant time period are used. N should be large enough to make sure that random variations are not interpreted as strategy changes.

To sum up: Each player uses only locally available and late (relevant) frequency information in the decision process.

3. COOPERATION OR CONFLICT IN THE TIMBER MARKET: A DUOPSONY DISCUSSION

Two sawmills buy timber from a large number of independent forest owners in an area. In this example we may interpret the forest industry input "timber" as a simple way to denote the content of what a forest industry company frequently buys, namely the right to harvest a forest stand including special agreements. Of course, the theory developed below should be relevant also in many other duopsony problems.

Every time a unit of timber is available, the forest owner receives sealed bids from the potential buyers. Clearly, this is a case where the buyers as a group may benefit from cooperation and low bids. The extra profit obtained via the low timber price may then be distributed between the buyers in some way. One example of a distribution principle is the Nash bargaining solution. Compare Nash [25] and [27].

In some cases, the strongest sawmill (in the sence of ability to survive high timber prices) may prefer not to cooperate and "destroy" the input market of the other sawmill via high bids. This way, both sawmills loose profits during some time period and the weakest sawmill is closed down. Then, the strongest sawmill has the option to use his monopsony power and to increase his profits even more than before via low timber prices.

The sawmill example contains two kinds of solutions:

In the cooperation case, we may expect the sawmills to calculate the timber price which maximizes the profit of the two sawmills as a group. Then they distribute the extra profit somehow within the group. Sometimes we may expect that the sawmills decide not only the timber price but also the distribution of the timber. The forest owners may not notice this cooperation directly. They may notice that all bids are low or that only one of the sawmills gives a bid on each unit of timber, or finally, that one sawmill gives a low bid and the other sawmill gives a very low bid on each timber unit. In the latest case, the very low bid is there just to hide the cooperation from the sellers. It does not affect the plan of the buyers anyway.

In the "timber price fight case", the timber price bids are high until one of the buyers leaves the market. Then the bids instantly fall and the low price level remains until increased competition appears.

Capen, Clapp and Campbell [7] have written a rather different but very interesting article on the optimal bidding level in another "resource sector", oil. In their problem, the greatest difficulty is to estimate the value of the oil field parcels. Clearly, it is easier to observe the trees above ground than the oil below the surface. The value estimations should be more correct in forestry.

Let us now turn to a third and more interesting case:

The buyers do not cooperate because they do not believe that the other buyers will keep an agreement. Maybe they are also aware that the government will discover market cooperation and punish cartels. Hence, the buyers act according to the law and sometimes deliver sealed bids.

When they decide to give a bid, they first have to inform themselves about the quality of the timber and other practical details. This activity is not costless. Then, they have to decide the level of the bid.

Of course, they can give a low bid and hope that the other sawmill will not give a higher bid. In that case, they will buy the timber cheaply. If they have bad luck, the other sawmill buys the timber with a higher bid and the only economic consequence of the activity is the cost of the investigation.

On the other hand, they may give a high bid and hope that the other sawmill will give a lower bid. The probability of obtaining the timber is of course higher in this case, but the price is also higher.

This last version of the game is interesting in several ways and the methodology to be used in the analysis is not obvious. Each player has in the example two different possible decisions: A high (H) or a low (L) bid. The players may be denoted by A and B.

If the sum of the total profit (the sum of the game) made by the two players is zero (or a constant), it is obvious that no cooperation will appear. Note that the profits of the players in this game are functions not only of the timber price but also of the resulting timber transactions and profits in the sawmills. The calculation of the profits conditional on the possible decision combinations is a complicated matter.

If the players know all the economic consequences for both players of all decision combinations exactly, then we can use the classical an well known two person zero sum game theory found in Luce and Raiffa [22] and Chiang [9]. The optimal strategies may turn out to be pure (only one decision) or mixed strategies for each player where a mixed strategy means that different decisions should be made with different probabilities. These optimal strategies, the probabilities of different decisions, can be calculated via linear programming.

If, which is frequently more likely, the sawmills have no (or very limited) information concerning the economic consequences in the other sawmill of different decision combinations, then it is not possible to calculate the optimal strategies directly via the traditional methods, such as linear programming.

Then, what should one do?

One obvious way for player A to deal with the problem is to observe and estimate the frequencies of the different decisions taken by player B. Then, player A maximizes his expected profit conditional on the decision frequencies of player B. Player A now plays the optimized strategy and player B observes and estimates the frequencies of the decisions made by player A. Most likely, the optimal strategy of player B is conditional on the strategy of player A. Hence B optimizes his strategy conditional on the latest frequency observations. Player B plays the new frequencies and player A soon changes strategy again.

Note that the profits of the two players are functions of many different physical and economical conditions except for the decision combination of the "game". These conditions include capacity constraints, sawn wood prices and labour costs, things that often change over time in a way which is not perfectly predictable by any player.

Thus, even if the economic consequences of different decision combinations are known exactly by both players, the sum of the game is constant and a linear programming solution can be calculated, it is not obvious that the derived strategies are optimal in later time periods. Compare the article by Kreps and Wilson [21] on sequential equilibria.

Robinson [31] has showed that if each player estimates the frequencies of the decisions made by the other player and then optimizes the own strategy conditional on the available information, then the solution converges to the solution obtained via linear programming.

In a changing environment, it is important to adapt rapidly, the question is only how rapidly. If A observes that B makes a particular decision 5 times in a row, – Is this an indication that B has started to play a pure strategy or is it just a quite possible sequence of observations with low probability? Maybe B still plays a mixed strategy with two different decisions, each with 50% probability?

–How many earlier observations of the other players decisions should be used in the frequency estimations? Of course, if only a few observations are used, the frequency estimations are more rapidly affected by a changing strategy used by B and A can optimize the strategy conditionally. On the other hand, if the number of observations of the decision sequence is too small, stochastic short term variations may be interpreted as strategy changes by the other player even if they are not. This, in turn, may lead to a new conditional strategy which is not optimal conditional on the true strategy of the other player.

4. THE ZERO SUM GAME AND IMPORTANT CONSEQUENCES OF THE IMPLICIT AND HIDDEN INFORMATION ASSUMPTION

Let us start by discussing the two person non cooperative game. In a very special case, the sum of the profits, the total pay off, of the two players is constant (it is not a function of the decision combination). Such games can be described and treated as "zero sum" games if the sum of the profits is reduced by the constant. The profit of one player is the loss of the other. Of course, we can not expect cooperation to appear is such a game. We have a simple game of conflict.

The definition of the zero sum game and the calculation of the Nash equilibrium via graphical methods and the simplex method can be found in Luce and Raiffa [22], Chiang [9] and many other standard books.

The two person zero sum game is often the only game that the student has time to learn. Unfortunately, the game has some very special properties of great importance to relevance and solvability:

In the zero sum game, each player knows exactly how the other player is affected by different decision combinations as long as he knows how he is affected by the combinations himself. This means that player A in principle can calculate exactly how player B will react under the assumption that B optimizes his behaviour conditional on the behaviour of A.

In a game which is not a zero sum game, player A does now know exactly how player B is affected by different decision combinations without a lot of special information concerning the (economic and/or maybe physical or biological) environment of player B.

Most importantly, if we make the model assumption that we have a zero sum game, this is not only a restrictive assumption concerning the pay off matrix which is seldom relevant. It is first of all an assumption that both players known very much (almost everything of relevance to economic decisions) about each other and that very simple methods can be used to determine how they will and should behave.

Now, let us turn to the pay off matrix.

A general two person non constant sum game will be analysed. As a very special case, we have the two person zero sum game discussed above. The Nash equilibria of this game can be calculated via the methods found in for instance Rasmusen [30]. We will not assume that the players know exactly how the other players are affected by different decision combinations. Each player however observes the frequencies of the decisions taken by the other player.

5. CONCEPTS, DEFINITIONS AND GENERAL EQUATIONS

Let $u(i,j)$ and $v(i,j)$ denote the pay off's that players A and B get respectively if player A makes decision i and player B makes decision j. $x(i)$ denotes the probability that player A makes decision i. $y(j)$ is the probability that player B makes decision j. We assume that each player has to possible decisions, 1 or 2. In order to simplify notation, we let $X = x(1)$ and $Y = y(1)$. Since the sums $(x(1) + x(2))$ and $(y(1) + y(2))$ must be equal to one, the coordinate (X, Y) in the two dimensional probability space reveals everything about the possibly mixed strategies used by the two players. Let the expected pay off's received by the players A and B be denoted by U and V.

These can be calculated as:

$$U = (u(1,1)Y + u(1,2)(1 - Y))X + (u(2,1)Y + u(2,2)(1 - Y))(1 - X) \tag{1}$$

$$V = (v(1,1)X + v(2,1)(1 - X))Y + (v(1,2)X + v(2,2)(1 - X))(1 - Y) \tag{2}$$

A continuously observes Y and B continuously observes X. Each player continuously adjusts his probability (X or Y) to the latest information. The marginal expected profits, dU/dX and dV/dY, can be calculated as:

$$dU/dX = (u(1,2) - u(2,2)) + (u(1,1) + u(2,2) - u(1,2) - u(2,1))Y \tag{3}$$

$$dV/dY = (v(2,1) - v(2,2)) + (v(1,1) + v(2,2) - v(1,2) - v(2,1))X \tag{4}$$

Obviously, notation can be simplified as:

$$dU/dX = m_1 + m_2 * Y \tag{5}$$

$$dV/dY = n_1 + n_2 * X \tag{6}$$

The adjustment speeds, dX/dt and dY/dt, are assumed to be proportional to the marginal expected profits of each player. Let us denote the proportionality constants by g and h.

$$dX/dt = g * dU/dX \tag{7}$$

$$dY/dt = h * dV/dY \tag{8}$$

This gives us a dynamical system where m_3, m_4, n_3 and n_4 are new parameters:

$$dX/dt = m_3 + m_4 * Y \tag{9}$$

$$dY/dt = n_3 + n_4 * X \tag{10}$$

The system is in equilibrium when $(X, Y) = (X_0, Y_0) = (-n_3/n_4, -m_3/m_4)$. Let us denote deviations from equilibrium by $(\mathbf{X}, \mathbf{Y})$ where $\mathbf{X} = X - X_0$ and $\mathbf{Y} = Y - Y_0$. Using this notation, we can rewrite the dynamical system as:

$$dX/dt = m_4 * \mathbf{Y} \tag{11}$$

$$dY/dt = n_4 * \mathbf{X} \tag{12}$$

Again, we simplify notation. Let $\mathbf{m} = m_4$, $\mathbf{n} = n_4$. Now, the time derivatives of the deviations from equilibrium, $\mathbf{dX/dt}$ and $\mathbf{dY/dt}$, can be written as:

$$\mathbf{dX/dt} = \mathbf{m_Y} \tag{13}$$

$$\mathbf{dY/dt} = \mathbf{n_X} \tag{14}$$

6. DYNAMICS AND A SIMPLE ZERO SUM GAME AS ILLUSTRATION

Let us determine the dynamical properties of the mixed strategy game, initially investigating a very special case, namely a two person zero sum game:

$$(u(1,1),\ u(1,2),\ u(2,1),\ u(2,2)) = (-1,\ 1,\ 1,\ -1) \tag{15}$$

$$(v(1,1),\ v(1,2),\ v(2,1),\ v(2,2)) = (1,\ -1,\ -1,\ 1) \tag{16}$$

Making use of the derived equations, we get:

$$dU/dX = +2 - 4Y \tag{17}$$

$$dV/dY = -2 + 4X \tag{18}$$

The dynamical system becomes

$$dX/dt = +2g - 4gY \tag{19}$$

$$dY/dt = -2h + 4hx \tag{20}$$

We find that the system is in equilibrium when $(X, Y) = (X_0, Y_0) = (1/2, 1/2)$. Deviations from equilibrium are denoted by $(\mathbf{X}, \mathbf{Y})$, where $\mathbf{X} = X - 1/2$ and $\mathbf{Y} = Y - 1/2$. We have:

$$\mathbf{dX/dt} = -4g\mathbf{Y} \tag{21}$$

$$\mathbf{dY/dt} = +4h\mathbf{X} \tag{22}$$

For simplicity, we initially assume that $g = h$. Let a new constant $\beta = 4g = 4h$. Now, we get:

$$\mathbf{dX/dt} = -\beta\mathbf{Y} \tag{23}$$

$$\mathbf{dY/dt} = +\beta\mathbf{X} \tag{24}$$

A family of solutions of the differential equations system, where z is an arbitrary positive constant, is:

$$\mathbf{X} = z \cos(\beta t) \tag{25}$$

$$\mathbf{Y} = z \sin(\beta t). \tag{26}$$

It is well known that $(\mathbf{X}, \mathbf{Y})$ will stay on the same distance (z) from the equilibrium for ever. The system does not converge to or diverge from the equilibrium. This is called a "center" in the theory of dynamical systems.

More generally, the adjustment speed coefficient g is not always the same as h. We can take care of this phenomenon via the introduction of one more constant, α. As long as g and h have different signs, we can always express the differential equations system as:

$$\mathbf{dX/dt} = -(1/\alpha)\beta\mathbf{Y} \tag{27}$$

$$\mathbf{dY/dt} = +(\alpha)\beta\mathbf{X} \tag{28}$$

The reader may verify that the following equations represent a family of solutions of our new differential equation system:

$$\mathbf{X} = z \cos(\beta t) \tag{29}$$

$$\mathbf{Y} = \alpha z \sin(\beta t) \tag{30}$$

Also this family of systems has periodic solutions that do not converge from or diverge to the equilibrium except in the sence that the orbit may be oval (α different from 1) and that the distance from $(\mathbf{X}, \mathbf{Y})$ to the equilibrium periodically increases and decreases. The interested reader may study the general principles of dynamical linear systems in the plane in Clark [10] or with a higher degree of detail in Braun [3].

7. THE NON ZERO SUM TIMBER MARKET DUOPSONY GAME

Consider the following pay off matrices:

$$(u(1,1),\ u(1,2),\ u(2,1),\ u(2,2)) = (0,\ 0,\ 0.8,\ -0.2) \tag{31}$$

$$(v(1,1),\ v(1,2),\ v(2,1),\ v(2,2)) = (1.8,\ 0.8,\ -0.2,\ 0.8) \tag{32}$$

These assumptions make sence in the following game: There are two sawmill firms more or less actively involved in the timber market area of the game and a large number of independent forest owners. The forest owners sometimes have timber to sell and they ask for sealed bids from the buyers. One of the sawmills, B is close to the area where the forests are located and has a big and very timber consuming sawmill. For these reasons, B always participates in the "bid games" and contributes with a low (L) or a high (H) bid. In every particular game, L means that the bid is consistent with a lower than average price and H represents a higher than average price.

Sawmill A, on the other hand, has a small sawmill which is located far away from the forests. Player A does not need very much timber and buys some of his timber in places closer to his mill. He has two possible decisions in each game: He does not participate at all (N) or he participates and gives a bid which is somewhere between the low and high bids given by sawmill B, a medium bid (M).

What are the possible outcomes?

If A does not participate, N, then B buys the timber irrespective of the bid level he choses (L or H). If A participates, M, then A buys the timber if B makes decision L and B buys the timber if B makes decision H.

What are the economic consequences of the different decision combinations? First of all, every player that participates investigates the object to decides the value. The cost of this investigation including other possibly necessary costs of participation is 0.2 (money units) in the example. The "shadow" value of the timber in the sawmill of player B is 2 units higher than the low bid of B and 1 unit higher than the high bid of the same player. Since A must undertake more expensive transportation etc., the shadow value of the timber in the enterprise of player A is only one unit higher than the medium bid sometimes given by player A.

The economic consequences of the different decision combinations may be presented in the following form:

Decision combination with numerical indices	**A gets**	**B gets**
(N and L) = (1, 1)	+0	+1.8
(N and H) = (1, 2)	+0	+0.8
(M and L) = (2, 1)	+0.8	− 0.2
(M and H) = (2, 2)	− 0.2	+0.8

The reader may verify that these consequences are consistent with the text and the pay off matrixes given in the introduction of this example. The assumptions imply:

$$dU/dX = +0.2 - 1Y \tag{33}$$

$$dV/dY = -1 + 2X \tag{34}$$

We find that the equilibrium is $(X_0, Y_0) = (0.5, 0.2)$. Let the adjustment speed coefficients be $g = h = 1$. Then, we get the following dynamical system of the deviation from equilibrium:

$$\mathbf{dX/dt} = -1\mathbf{Y} \tag{35}$$

$$\mathbf{dY/dt} = +2\mathbf{X} \tag{36}$$

Let us calculate the parameters α and β to determine the time path of the system:

$$-(1/\alpha)\beta \quad = -1 \tag{37}$$

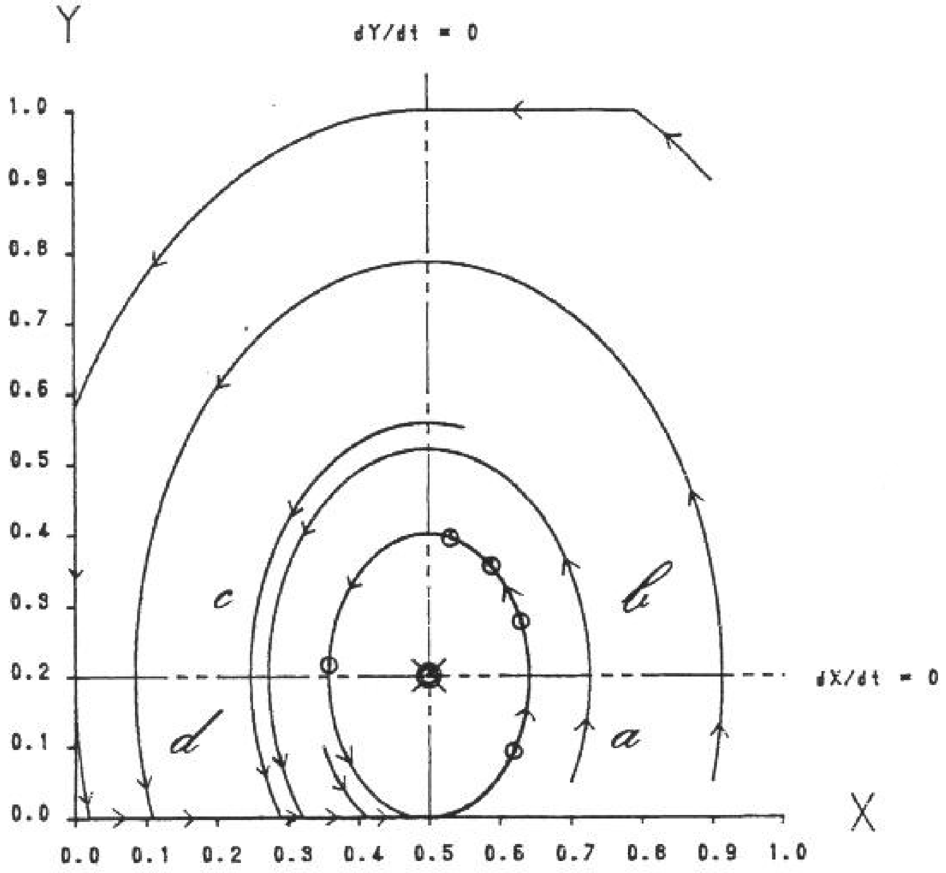

FIGURE 1 Decision Probability Trajectories $\{u_{11}, u_{12}, u_{21}, u_{22} = \{0, 0, .8, -.2\}$ $\{v_{11}, v_{12}, v_{21}, v_{22}\} = \{1.8, .8, -.2, .8\}$.

$$\alpha\beta \quad = +2 \tag{38}$$

The solution of this system is $\alpha = \beta = \sqrt{2}$ or $\alpha = \beta = -\sqrt{2}$. The system follows a pair of equations that belong to the following family:

$$\mathbf{X} = z \cos(\beta t) \tag{39}$$

$$\mathbf{Y} = \alpha z \sin(\beta t) \tag{40}$$

where $(\mathbf{X}, \mathbf{Y})$ denotes deviations from the equilibrium $(X_0, Y_0) = (0.5, 0.2)$. Note that the time path of this system is the same if $\alpha = \beta = \sqrt{2}$ as if $\alpha = \beta = -\sqrt{2}$. If we select the negative value, however, time moves backwards in our equations. Later on, when we analyse the effects of the probability constraints on the dynamics of the strategy combination, time becomes irreversible in several cases.

As long as the constraints are not affecting the time path, time may go backwards. Compare the interesting the related discussion in Prigogine and Stengers [28]. Finally:

$$(X, Y) = ((0.5 + z \cos(\beta t)), \quad (0.2 + \alpha z \sin(\beta t))) \tag{41}$$

The reader can verify that the computer generated graphical solution in Figure 1 supports this conclusion as long as the equations do not force the system to leave the feasible set.

So, it seems that we should expect the two players to move around for ever in an oval orbit that in a special case is a circle, continuously adjusting the probabilities X and Y. However, as we know, (X, Y) may not leave the unit square in the positive quadrant. A natural reformulation of the original dynamical system is:

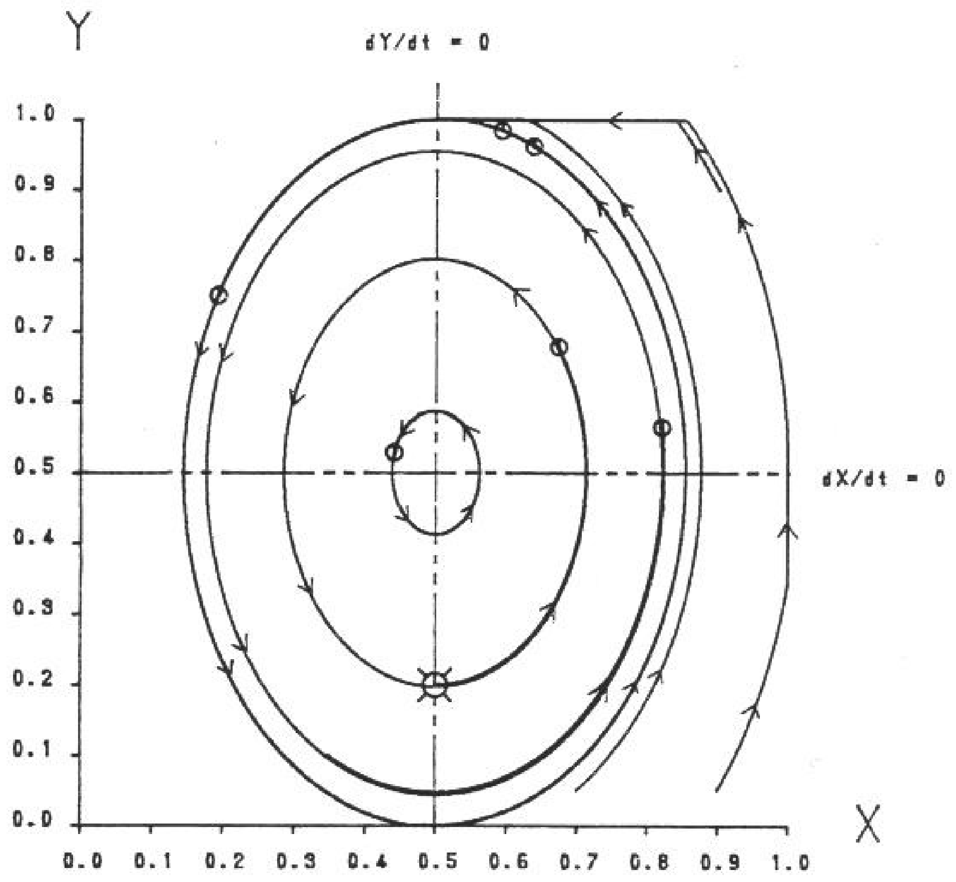

FIGURE 2 Decision Probability Trajectories $\{u_{11}, u_{12}, u_{21}, u_{22} = \{0, 0, .5, -.5\}$ $\{v_{11}, v_{12}, v_{21}, v_{22}\} = \{1.5, 5, -.5, .5\}$.

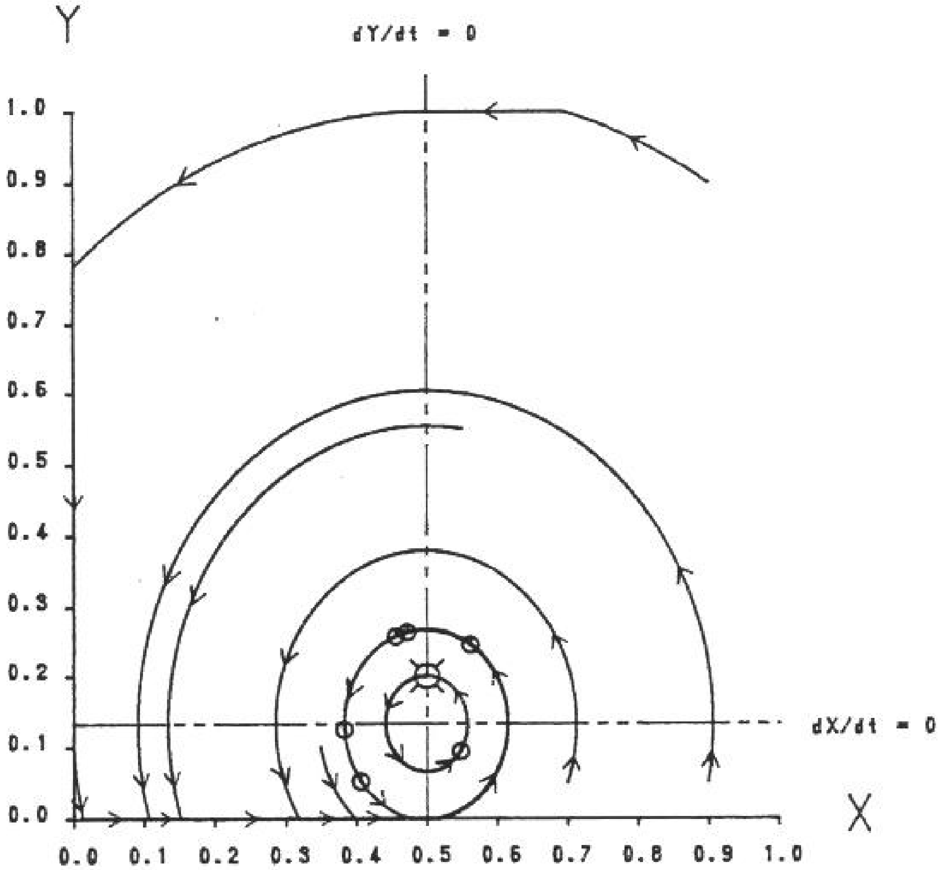

FIGURE 3 Decision Probability Trajectories $\{u_{11}, u_{12}, u_{21}, u_{22} = \{0, 0, 1.3, -.2\}$ $\{v_{11}, v_{12}, v_{21}, v_{22}\} = \{1.8, .8, -.2, .8\}$.

Differential equation	**Condition**	
$dX/dt = 0$	if $X = 0$ and $g * dU/dX < 0$	
$dX/dt = g * dU/dX$	if $X = 0$ and $g * dU/dX \geq 0$	
$dX/dt = g * dU/dX$	if $0 < X < 1$	(42)
$dX/dt = 0$	if $X = 1$ and $g * dU/dX > 0$	
$dX/dt = g * dU/dX$	if $X = 1$ and $g * dU/dX \leq 0$	
$dY/dt = 0$	if $Y = 0$ and $h * dV/dY < 0$	
$dY/dt = h * dV/dY$	if $Y = 0$ and $h * dV/dY \geq 0$	
$dY/dt = h * dV/dY$	if $0 < Y < 1$	(43)
$dY/dt = 0$	if $Y = 1$ and $h * dV/dY > 0$	
$dY/dt = h * dV/dY$	if $Y = 1$ and $h * dV/dY \leq 0$	

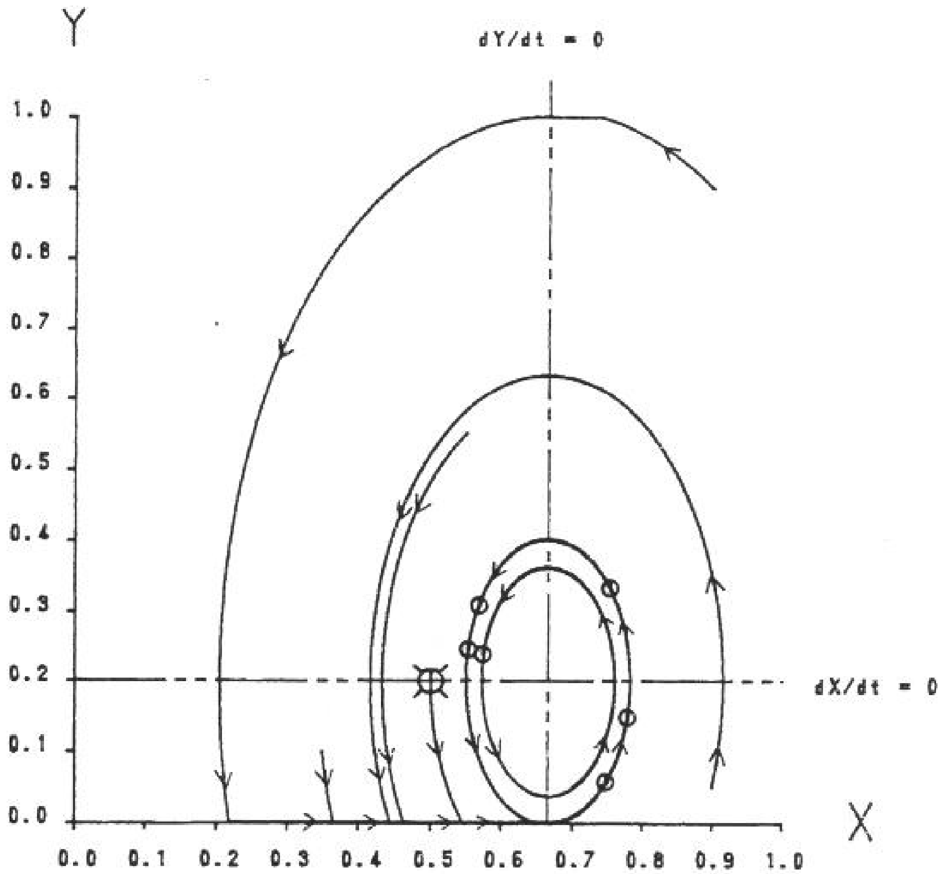

FIGURE 4 Decision Probability Trajectories $\{u_{11}, u_{12}, u_{21}, u_{22}\} = \{0, 0, .8, -.2\}$ $\{v_{11}, v_{12}, v_{21}, v_{22}\} = \{2.8, 1.8, -.2, 1.8\}$.

The equations mean that X and Y continuously are adjusted in the "most profitable" directions as long as the movement does not take the system out of the feasible set, the unit square in the first quadrant. The graphical result, the trajectories found in Figure 1., show the time path of the strategy combination (X, Y). The time path of the system has been plotted for the same 6 initial conditions in the Figures 1–4 The positions of the system after 3000 time units are marked with small circles. The time units are short in relation to the adjustment speed coefficients. g and h have been given the value 0.002.

We can make the followng observations in Figure 1:

Region a

Sawmill A frequently does not participate, $X > 0.5$, and sawmill B almost always gives a high bid, $Y < 0.2$.

B most of the time gets the timber even if only the low bid is given. Hence, B increases the frequency of the low bid. A almost never gets the timber even if A participates. Hence, A decreases the frequency of participation even more. The system is moved upwards and to the right and soon reaches Region B.

Region b

Sawmill A frequently does not participate, $X > 0.5$, and sawmill B frequently gives a low bid, $Y > 0.2$.

B most of the time gets the timber even if only the low bid is given. Hence, B increases the frequency of the low bid. When A participates, he often gets the timber and hence finds it profitable to participate more often. The system moves upward and to the left, finally reaching Region c.

Region c

A often participates, $X < 0.5$, and B often gives low bids, $Y > 0.2$.

Since B frequently gives low bids, A finds it profitable to participate more often. B discovers that he has to increase the frequency of high bids in order to get any timber since A participates most of the time with his medium bid. The system is moved down to the left, reaching Region d.

Region d

A participates most of the time, $X < 0.5$, and B gives high bids most of the time, $Y < 0.2$.

A realizes that he does not buy timber very often anymore but that he has to pay the expensive timber investigations anyway as long as he participates. He decides to participate more seldom. B still finds that A participates rather often and that he can increase the amount of timber he buys if he increases the high bid frequency. The system is moved down to the right and finally reaches Region a again.

Observation 1

The Nash equilibrium solution, $(X, Y) = (0.5, 0.2)$, will **never** be reached unless that happens to be the initial condition of the system.

Observations 2
If the system follows a trajectory, an orbit or a center, that passes through the four different regions without touching the boundary of the feasible area, then the system will follow this orbit for ever.

Observations 3
If the system follows a trajectory that somewhere touches the boundary of the feasible area, then the system will follow the boundary for some time. Finally, the system will start to follow an attractor, a "center", for ever. This attractor will be the largest center than can be constructed around the equilibrium, without touching the boundaries, which is consistent with the unconstrained differential equations. Note that most of the small circles in the Figures 1 – 4 have been trapped for ever in the respective attractors.

8. DYNAMIC SENSITIVITY ANALYSIS OF THE TIMBER MARKET GAME

Now, we will partially modify the assumptions concerning the economic environment of the two firms in the game.

Case 1
Assumptions: According to the duopsony game formulated above.
Equilibrium: $(X, Y) = (0.5, 0.2)$
Illustration: Figure 1. The equilibrium of Case 1 is marked with a star and used as one of the initial conditions in Cases 1 – 4. This way it is possible to see how a system, initially in equilibrium, dynamically is affected by changes in the parameters.

Case 2
Assumptions: As in Case 1 except for that the investigation cost in the game is 0.5 units for every participant.
Equilibrium: $(X, Y) = (0.5, 0.5)$
Illustration: Figure 2
Observations: In equilibrium, A will participate in the game as frequently as in Case 1 but B will on the average give lower bids. Trajectory from the equilibrium of Case 1: Player B is happy with his

old frequency 0.2 as long as A goes on with his frequency 0.5. In 50% of the games, B increases his profit by 1 unit via decision 2 and in 50% of the games, he decreases the profit by the same amount. However, in the new cost situation, A finds it too expensive to participate every second time. He does not get the timber in more than 20% of the games where he decides to participate, namely when B gives low bids. If he gets the timber, (which seldom happens), then he earns no more than what he looses if he does not get the timber (which is the more common result). Hence, A decreases his participation and starts a more or less circular orbit of the system which almost always will be located above the old equilibrium. The qualitative behaviour of the system will be the same as in Case 1. but the new equilibrium will be found above the old equilibrium. The trajectory will take the system to the old equilibrium one time per period but it will never stay there.

Case 3

Assumptions: As in Case 1 except for that the variable unit production cost decreases in sawmill A and/or the price of the sawn wood produced in sawmill A increases. The net profit of player A increases by 0.5 units compared to Case 1 if A gets the timber.

Equilibrium: $(X, Y) = (0.5,\ 0.13)$

Illustration: Figure 3

Observations: In equilibrium, A will participate in the game as often as in Case 1 but B will more often give high bids.

Trajectory from the equilibrium of Case 1: Player A starts the movement from the old equilibrium and the system will most of the time be found below the old equilibrium. Player B is satisfied with the old equilibrium. It can be shown that B can not increase his expected pay off through changes in his probability Y. On the other hand, it is now more important to A to get the timber than before. He finds it profitable to participate more often than in Case 1 as long as B continues his strategy. Hence, A pushes the system to the left. Suddenly, B finds that he has to increase his frequency of high bids in order to optimize his expected pay off. The system is moved down to the left. Soon, A notices that the expected bid of B is higher than before. A does not get very much timber anymore. B continues to increase his high bid frequency. A participates less often and the system moves down to the right. Now, B plays high bids all the time and A decreases his participation even more. B notices this and discovers that he gets most of the timber also when he plays low bids. The system is moved upwards to the right. When the frequency of low bids is high enough, A decides to participate more often. The system goes back to the equilibrium of Case 1.

Case 4

Assumptions: As in Case 1 except for that the variable unit production cost decreases in sawmill B and/or the price of the sawn wood produced in sawmill B increases. Hence, the net profit of player B increases by 1 unit if B gets the timber compared to Case 1.

Equilibrium: $(X, Y) = (0.67,\ 0.2)$

Illustration: Figure 4

Observations: In equilibrium, A participates less frequently than in Case 1 and B gives the same expected bid level as in Case 1.

Trajectory from the equilibrium of Case 1: Player A is satisfied in the equilibrium of Case 1 but B wants to increase his proportion of games where he wins the timber. Thus, B moves the system downwards, increasing his frequency of high bids. Because the probability of winning the timber is lower than before, A reduces his participation. The system moves down and to the right. In the illustration, B reaches the boundary where he plays the high bid all of the time. A continues to decrease his participation and soon the system is "trapped" in the oval attractor which is tangent to the boundary of the feasible set. A will always participate less frequently and B will on the average give the same bid level as in the equilibrium of Case 1.

9. PERIODIC SOLUTIONS AND THE PROPERTIES OF THE PAY OFF MATRIX

So far, we have just assumed the values of the pay off coefficients. In the suggested examples, the solutions have been constrained probability orbits. Let us investigate the pay off conditions to be expected if we observe periodic decision probability changes.

We already know that $(X_0, Y_0) = (-n_3/n_4, -m_3/m_4) = (-n_1/n_2, -m_1/m_2)$. If we want to get cyclical solutions, we want the signs of n_4

and m_4 to be different (Compare Equations (27) – (30)). This means that the signs of n_2 and m_2 are dfferent as long as both players have strictly prositive adjustment speed coefficients. Let us here make the restriction that n_2 is positive and that m_2 is negative. Since the names of the players have not been decided, this assumption is not restrictive. Since we want (X_0, Y_0) to be found inside the feasible area, we have:

$$(0 < X_0 < 1) \text{ implies}: \ 0 < -n_1/n_2 < 1 \tag{44}$$

$$(0 < Y_0 < 1) \text{ implies}: \ 0 < -m_1/m_2 < 1 \tag{45}$$

Here, notation is again simplified and u_{ij} and v_{ij} denote $u(i,j)$ and $v(i,j)$ respectively.

$$(X_0 < 1) \text{ implies}: \ (v_{22} - v_{21})/(v_{11} + v_{22} - v_{12} - v_{21}) < 1 \tag{46}$$

Since $n2 > 0$, we can write:

$$(v_{22} - v_{21}) < (v_{11} + v_{22} - v_{12} - v_{21}) \tag{47}$$

and finally:

$$0 < (v_{11} - v_{12}) = z1 \tag{48}$$

This means that player B prefers the decision combination (1,1) to (1,2).

$$(0 < X_0) \text{implies}: 0 < (v_{22} - v_{21})/(v_{11} + v_{22} - v_{12} - v_{21}) \tag{49}$$

Let us rewrite this expression:

$$0 < (v_{22} - v_{21})/(v_{22} - v_{21} + z_1) \tag{50}$$

Since $z_1 > 0$ and we want the condition to hold for all positive values of $z_1, (v_{22} - v_{21}) > 0$. This means that player B prefers decision combination (2,2) to (2,1).

We conclude these findings the following way: Player B prefers to play the same decision index as player A. If A plays $i = 1$, then B wants to play $j = 1$. If A plays $i = 2$, then B wants to play $j = 2$. In other words, Player B prefers to be on the "main diagonal", (1,1) and (2,2) of the decision matrix.

$$(Y_0 < 1) \text{ implies : } (u_{22} - u_{12})/(u_{11} + u_{22} - u_{12} - u_{21}) < 1 \qquad (51)$$

We initially assumed that $m2 < 0$. Thus:

$$(u_{22} - u_{12}) > (u_{11} + u_{22} - u_{12} - u_{21}) \qquad (52)$$

After simplification, we get:

$$0 > (u_{11} - u_{21}) = z_2 \qquad (53)$$

which means that A prefers decision combination (2,1) to (1,1).

$$(0 < Y_0) \text{ implies : } 0 < (u_{22} - u_{12})/(u_{11} + u_{22} - u_{12} - u_{21}) \qquad (54)$$

This can be rewritten as:

$$0 < (u_{22} - u_{12})/(u_{22} - u_{12} + z_2) \qquad (55)$$

Since $z_2 < 0$ and we want the condition to hold for all negative values of z_2, this means that $u_{22} < u_{12}$.

To sum up: Player A prefers (2,1) to (1,1) and (1, 2) to (2, 2). In other words, Player A prefers to be off the main diagonal of the decision matrix.

Observation 4

We sometimes have a game where conditions c1. and c2. are satisfied:

c1. Each player optimizes his expected pay off via a mixed strategy conditionally on the decision frequencies of the other player. In the mixed strategies, every decision should have a strictly positive probability.

c2. The differential equation system governing the simultaneous optimal adjustments of the decision frequencies of the two players give cyclical solutions, sine and cosine functions.

c1 and c2 are consistent with a situation where the different players prefer decision combinations on different diagonals of the decision matrix. One of the players prefers to adjust the system to the main diagonal and the other player prefers to adjust the system to the other diagonal.

10. DISCUSSION

This paper contains a simple treatment of the dynamics of the non constant sum (and constant sum as a special case) game where the players only make use of local information and continuous decision frequency observations.

It is found that a large number of possible initial conditions make the decision probability combination follow a special form of attractor and that centers can be expected to appear in typical games. The probability that the Nash equilibrium will be the solution is almost zero.

Conditions relating the decision frequencies in equilibrium and the cyclical but constrained nature of the solution to the pay off matrix properties have been derived.

A dynamic duopsony timber market game has been defined and studied. In particular, the trajectories of the decision probability combination, were investigated.

In a special analysis, it was assumed that the system initially was in equilibrium and that the pay off coefficients for different reasons were changed. After each parameter change, the system got a new equilibrium but did not converge to this. The system started a new orbit, a center, around the new equilibrium.

Real world games are complicated. Hopefully, the reader has found the analysis in this paper to be a step in the right direction. Useful theories should approach reality in the long run. When we find a game in reality where the players use mixed strategies and change the frequencies over time, we have an indication that the present theory is relevant.

Note

The author has recently become aware that also Shapley (1964) discovered that periodic solutions are possible in two person games. Shapley, however, assumed that the players choose a strategy that would yield the optimum result if employed against all past choices of their opponents. He claimed that the problems with cycles begin if the players have three or more strategies available. In the present paper, only the latest (and presently relevant) frequency information is used, only two possible decisions per player are necessary to generate the cyclical results, the frequency adjustment process is different and several properties of the cyclical solutions are different.

Acknowledgements

I gratefully acknowledge research grants from Brattåsstiftelsen för Skogsvetenskaplig Forskning, Cellulosaindustriens Stiftelse för Teknisk och Skoglig Forskning samt Utbildning, Domänverket, Lars-Erik Thunholms Stiftelse för Främjande av Vetenskaplig Forskning and Skogs och Jordbrukets Forskningsråd. Professor Sjur D. Flåm, Department of Economics, Bergen, Norway, made me discover Cournot. Master of Science Peter Fredman, The Swedish University of Agricultural Sciences, Department of Forest Economics, checked my formulae.

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