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Optimal Resource Control Model

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General continuous time optimal control model of a forest resource, comparative dynamics and CO2 storage consideration effects

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First some

General optimal control theory

$$\max_{u(s)} \left\{ J = \int_t^T F\left(x, u, s\right) ds + S\left(x(T), T\right) \right\}$$

$$u(s) \in \Omega(s)$$

$$\dot{x} = f\left(x, u, t\right) \qquad x(0) = x_0$$

$$V(x,t)=\max_{u(s)}\int_t^T F\left(x(s),u(s),s\right)ds+S\left(x(T),T\right)$$

$$u(s)\in\Omega(s)$$

$$\dot{x}=\frac{\partial x}{\partial s}=f(x(s),u(s),s)\qquad x(0)=x_0$$

$$V(x,t) = \max_{u(s)} \left\{ \int_{t}^{t+\partial t} F\left(x(s), u(s), s\right) ds + V\left(x(t+\partial t), t+\partial t\right) \right\} + o(\partial t)$$

$$u(s) \in \Omega(s)$$

$$V(x,t) = \max_{u(t)} \left\{ F\left(x(t), u(t), t \right) \partial t + V(x(t+\partial t), t+\partial t) \right\} + o(\partial t)$$

$$u(t) \in \Omega(t)$$

Taylor expansion:

$$V(x(t+\partial t), t+\partial t) = V(x,t) + \left(V_x(x,t)\dot{x} + V_t(x,t) \right)\partial t + o(\partial t)$$

$$V(x(t+\partial t), t+\partial t) = V(x,t) + V_x(x,t) f(x,u,t)\partial t + V_t(x,t)\partial t + o(\partial t)$$

$$V(x,t) = \max_{u(t)} \left\{ F\left(x,u,t\right)\partial t + V(x,t) + V_x(x,t) f(x,u,t)\partial t + V_t(x,t)\partial t \right\} + o(\partial t)$$

$$u(t) \in \Omega(t)$$

$$0 = \max_{u(t)} \left\{ F\left(x, u, t\right) + V_x(x, t) f(x, u, t) + V_t(x, t) \right\} \partial t + o(\partial t)$$

$$u(t) \in \Omega(t)$$

$$0 = \max_{u(t)} \left\{ F\left(x,u,t\right) + V_x(x,t) f(x,u,t) + V_t(x,t) \right\} + \frac{o(\partial t)}{\partial t}$$

$$u(t) \in \Omega(t)$$

$$\lim_{\partial t \to 0} \frac{o(\partial t)}{\partial t} = 0$$

$$0 = \max_{u(t)} \left\{ F\left(x, u, t\right) + V_x(x,t) f(x,u,t) + V_t(x,t) \right\}$$

$$u(t) \in \Omega(t) \quad ; \quad V(x,T) = S(x,T)$$

$$\lambda(t) = V_x(x^*(t), t)$$

$$H = H(x, u, V_x, t) = F(x, u, t) + V_x(x, t) f(x, u, t)$$

$$H = H(x, u, \lambda, t) = F(x, u, t) + \lambda(t) f(x, u, t)$$

$$0 = \max_{u(t)} \left\{ H(x, u, V_x, t) + V_t(x, t) \right\}$$

$$u(t) \in \Omega(t)$$

$V_t(x,t)$ **is not a function of** u

Hence, we may exclude $V_t(x,t)$ **from the optimization.**

$$0 = \left\{ \max_{u(t)} H(x, u, V_x, t) \right\} + V_t(x,t)$$

$$u(t) \in \Omega(t)$$

$x(t)$ **is optimized via** $u(t)$

We assume an unconstrained local maximum.

$$H_x + V_{tx} = 0$$

$$H_x = -V_{tx} = -V_{xt}$$

$$H_x = -\dot{V}_x$$

The adjoint equation:

$$H_x = -\dot{\lambda}$$

Terminal boundary condition:

$$\lambda(T) = \left.\frac{\partial S\left(x,T\right)}{\partial x}\right|_{x=x(T)} = S_x\left(x(T),T\right)$$

$$\begin{cases} \dot{x} = H_{\lambda} \quad , \quad x(0) = x_0 \\ \dot{\lambda} = -H_X \quad , \quad \lambda(T) = S_x\left(x(T), T\right) \end{cases}$$

The maximum principle:
Necessary conditions for

$$u^*$$

to be an optimal control:

$$\begin{cases} \dot{x}^* = f(x^*, u^*, t) \quad , \quad x^*(0) = x_0 \\ \dot{\lambda} = -H_X(x^*(t), u^*(t), \lambda(t), t) \quad , \quad \lambda(T) = S_x\left(x^*(T), T\right) \\ H(x^*(t), u^*(t), \lambda(t), t) \geq H(x^*(t), u(t), \lambda(t), t) \quad \forall u \in \Omega(t), t \in [0, T] \end{cases}$$

Now, a more specific

Optimal control model

$$\max\left\{ J = \int_{t_1}^{t_2} e^{-rt}\left(\left(f_1 + f_2 t\right)x + \left(k_1 + k_2 t\right)u + k_3 u^2\right)dt \right\}$$

- $$\dot{x} = g_0 + g_1 x + g_2 t - u$$

$$x(t_1) = x_1; \quad x(t_2) = x_2$$

$$H = e^{-rt}\left(\left(f_1 + f_2 t\right)x + \left(k_1 + k_2 t\right)u + k_3 u^2\right) + \lambda\left(g_0 + g_1 x + g_2 t - u\right)$$

$$\frac{\partial H}{\partial u} = e^{-rt}\left(k_1 + k_2 t + 2k_3 u\right) - \lambda = 0$$

$$\frac{\partial H}{\partial x} = e^{-rt}\left(f_1 + f_2 t\right) + g_1 \lambda$$

Determination of the time derivative of λ

via the adjoint equation:

$$\frac{\partial H}{\partial x} = -\dot{\lambda}$$

$$e^{-rt}\left(f_1 + f_2 t\right) + g_1 \lambda = -\dot{\lambda}$$

$$\dot{\lambda} = -g_1 \lambda - e^{-rt}\left(f_1 + f_2 t\right)$$

$$\left[\frac{\partial H}{\partial u} = e^{-rt}\left(k_1 + k_2 t + 2k_3 u\right) - \lambda = 0\right] \rightarrow \left[\lambda = e^{-rt}\left(k_1 + k_2 t + 2k_3 u\right)\right]$$

$$\dot{\lambda} = -g_1 e^{-rt}\left(k_1 + k_2 t + 2k_3 u\right) - e^{-rt}\left(f_1 + f_2 t\right)$$

$$\dot{\lambda} = e^{-rt}\left(-g_1\left(k_1 + k_2 t + 2k_3 u\right) - f_1 - f_2 t\right)$$

$$\dot{\lambda} = e^{-rt}\left(-\left(g_1 k_1 + f_1\right) - \left(g_1 k_2 + f_2\right)t - 2g_1 k_3 u\right)$$

Determination of the time derivative of λ **via** $\frac{\partial H}{\partial u} = 0$

$$\left[\frac{\partial H}{\partial u} = 0\right] \rightarrow \left[\lambda = e^{-rt}\left(k_1 + k_2 t + 2k_3 u\right)\right]$$

$$\dot{\lambda} = -re^{-rt}\left(k_1 + k_2 t + 2k_3 u\right) + e^{-rt}\left(k_2 + 2k_3 \dot{u}\right)$$

$$\dot{\lambda} = e^{-rt}\left(-r\left(k_1 + k_2 t + 2k_3 u\right) + k_2 + 2k_3 \dot{u}\right)$$

$$\dot{\lambda} = e^{-rt}\left(\left(k_2 - k_1 r\right) - k_2 rt - 2k_3 ru + 2k_3 \dot{u}\right)$$

The time derivative of λ **as determined via** $\dfrac{\partial H}{\partial u} = 0$

must equal the time derivative of λ

as determined via the adjoint equation:

$$\left(k_2 - k_1 r\right) - k_2 rt - 2k_3 ru + 2k_3 \dot{u} = -\left(g_1 k_1 + f_1\right) - \left(g_1 k_2 + f_2\right)t - 2g_1 k_3 u$$

$$2k_3 \dot{u} + \left(2g_1 k_3 - 2k_3 r\right)u \;=\; \left(k_1 r - k_2 - g_1 k_1 - f_1\right) + \left(k_2 r - g_1 k_2 - f_2\right)t$$

$$2k_3 \dot{u} - 2k_3\left(r - g_1\right)u \;=\; \left(k_1\left(r - g_1\right) - k_2 - f_1\right) + \left(k_2\left(r - g_1\right) - f_2\right)t$$

Definition:

$$r_2 = r - g_1$$

Observation:
This differential equation can be used to determine the optimal control function $u(t)$

$$\dot{u} - r_2 u = \frac{1}{2k_3}\left(k_1 r_2 - k_2 - f_1\right) + \frac{1}{2k_3}\left(k_2 r_2 - f_2\right)t$$

Solution of the homogenous differential equation:

$$\dot{u}_h - r_2 u_h = 0$$

Let $u_h = Ae^{mt}$

$$(m - r_2)\,Ae^{mt} = 0\,;\ Ae^{mt} \neq 0$$

$$m = r_2$$

$$u_h(t) = Ae^{r_2 t}$$

Determination of the particular solution:

$$\dot{u}_p - r_2 u_p = \frac{1}{2k_3}\left(k_1 r_2 - k_2 - f_1\right) + \frac{1}{2k_3}\left(k_2 r_2 - f_2\right)t$$

Let $u_p = z_1 + z_2 t$

$$z_2 - r_2\left(z_1 + z_2 t\right) = \frac{1}{2k_3}\left(k_1 r_2 - k_2 - f_1\right) + \frac{1}{2k_3}\left(k_2 r_2 - f_2\right)t$$

$$\left(z_2 - z_1 r_2\right) - z_2 r_2 t = \frac{1}{2k_3}\left(k_1 r_2 - k_2 - f_1\right) + \frac{1}{2k_3}\left(k_2 r_2 - f_2\right)t$$

$$-z_2 r_2 = \frac{1}{2k_3}\left(k_2 r_2 - f_2\right)$$

$$z_2 = \frac{1}{2k_3}\left(\frac{f_2}{r_2} - k_2\right)$$

$$z_2 - z_1 r_2 = \frac{1}{2k_3}\left(k_1 r_2 - k_2 - f_1\right)$$

$$\frac{1}{2k_3}\left(\frac{f_2}{r_2} - k_2\right) - z_1 r_2 = \frac{1}{2k_3}\left(k_1 r_2 - k_2 - f_1\right)$$

$$-z_1 r_2 = \frac{1}{2k_3}\left(k_1 r_2 - k_2 - f_1\right) - \frac{1}{2k_3}\left(\frac{f_2}{r_2} - k_2\right)$$

$$z_1 = \frac{1}{2k_3}\left(\frac{f_1}{r_2} + \frac{f_2}{\left(r_2\right)^2} - k_1\right)$$

$$u(t) = u_h(t) + u_p(t)$$

Conclusion:

The optimal control function is:

$$u(t) = Ae^{r_2 t} + z_1 + z_2 t$$

Determination of the optimal stock path equation:

$$\dot{x} = g_0 + g_1 x + g_2 t - u$$

The optimal control function is introduced in the differential equation of the stock path equation:

$$\dot{x} = g_0 + g_1 x + g_2 t - \left(A e^{r_2 t} + z_1 + z_2 t \right)$$

As a result, the differential equation of the optimal stock path equation is:

$$\dot{x} - g_1 x = (g_0 - z_1) + (g_2 - z_2)t - Ae^{r_2 t}$$

Solution of the homogenous differential equation:

$$\dot{x}_h - g_1 x_h = 0$$

Let $$x_h = Be^{nt}$$

$$\left(n - g_1\right) Be^{nt} = 0\,;\, Be^{nt} \neq 0$$

$$n = g_1$$

$$x_h(t) = Be^{g_1 t}$$

Determination of the particular solution:

$$\dot{x}_p - g_1 x_p = \left(g_0 - z_1\right) + \left(g_2 - z_2\right)t - Ae^{r_2 t}$$

Let: $x_p = c_0 + c_1 t + c_2 e^{r_2 t}$

$$\dot{x}_p = c_1 + r_2 c_2 e^{r_2 t}$$

$$c_1 + r_2 c_2 e^{r_2 t} - g_1\left(c_0 + c_1 t + c_2 e^{r_2 t}\right) = \left(g_0 - z_1\right) + \left(g_2 - z_2\right)t - Ae^{r_2 t}$$

$$\left(c_1 - c_0 g_1\right) - c_1 g_1 t + c_2\left(r_2 - g_1\right)e^{r_2 t} = \left(g_0 - z_1\right) + \left(g_2 - z_2\right)t - Ae^{r_2 t}$$

$$-c_1 g_1 = g_2 - z_2$$

$$c_1 = \frac{z_2 - g_2}{g_1}$$

$$c_1 - c_0 g_1 = g_0 - z_1$$

$$\frac{z_2 - g_2}{g_1} - c_0 g_1 = g_0 - z_1$$

$$-c_0 g_1 = g_0 - z_1 - \frac{z_2 - g_2}{g_1}$$

$$c_0 = \frac{1}{g_1}\left(z_1 + \frac{z_2 - g_2}{g_1} - g_0 \right)$$

$$c_2\left(r_2 - g_1\right) = -A$$

$$c_2 = \frac{A}{g_1 - r_2} \quad \Leftrightarrow \quad A = \left(g_1 - r_2\right)c_2$$

$$x(t) = x_h(t) + x_p(t)$$

Conclusion:

The optimal stock path equation is:

$$x(t) = Be^{g_1 t} + c_0 + c_1 t + c_2 e^{r_2 t}$$

Determination of (A, B) **via the initial and terminal conditions**

$$x(t_1) = x_1; \quad x(t_2) = x_2$$

$$x(t) = Be^{g_1 t} + c_0 + c_1 t + A\frac{e^{r_2 t}}{g_1 - r_2}$$

$$\begin{cases} x_1 = Be^{g_1 t_1} + c_0 + c_1 t_1 + A\dfrac{e^{r_2 t_1}}{g_1 - r_2} \\ x_2 = Be^{g_1 t_2} + c_0 + c_1 t_2 + A\dfrac{e^{r_2 t_2}}{g_1 - r_2} \end{cases}$$

$$\begin{cases} \dfrac{e^{r_2 t_1}}{g_1 - r_2} A + e^{g_1 t_1} B = x_1 - c_0 - c_1 t_1 \\ \dfrac{e^{r_2 t_2}}{g_1 - r_2} A + e^{g_1 t_2} B = x_2 - c_0 - c_1 t_2 \end{cases}$$

$$\begin{bmatrix} \dfrac{e^{r_2 t_1}}{g_1 - r_2} & e^{g_1 t_1} \\ \dfrac{e^{r_2 t_2}}{g_1 - r_2} & e^{g_1 t_2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} x_1 - c_0 - c_1 t_1 \\ x_2 - c_0 - c_1 t_2 \end{bmatrix}$$

$$|D| = \begin{vmatrix} \dfrac{e^{r_2 t_1}}{g_1 - r_2} & e^{g_1 t_1} \\ \dfrac{e^{r_2 t_2}}{g_1 - r_2} & e^{g_1 t_2} \end{vmatrix} = \frac{e^{r_2 t_1}}{g_1 - r_2} e^{g_1 t_2} - \frac{e^{r_2 t_2}}{g_1 - r_2} e^{g_1 t_1}$$

$$|D| = \frac{e^{r_2 t_1} e^{g_1 t_2} - e^{r_2 t_2} e^{g_1 t_1}}{g_1 - r_2}$$

$$|D| = \frac{e^{(r_2 t_1 + g_1 t_2)} - e^{(r_2 t_2 + g_1 t_1)}}{g_1 - r_2}$$

$$A = \frac{\begin{vmatrix} x_1 - c_0 - c_1 t_1 & e^{g_1 t_1} \\ x_2 - c_0 - c_1 t_2 & e^{g_1 t_2} \end{vmatrix}}{|D|}$$

$$A = \frac{\left(x_1 - c_0 - c_1 t_1\right) e^{g_1 t_2} - \left(x_2 - c_0 - c_1 t_2\right) e^{g_1 t_1}}{\left|D\right|}$$

$$B = \frac{\begin{vmatrix} \dfrac{e^{r_2 t_1}}{g_1 - r_2} & x_1 - c_0 - c_1 t_1 \\ \dfrac{e^{r_2 t_2}}{g_1 - r_2} & x_2 - c_0 - c_1 t_2 \end{vmatrix}}{|D|}$$

$$B = \frac{\left(\frac{e^{r_2 t_1}}{g_1 - r_2}\right)\left(x_2 - c_0 - c_1 t_2\right) - \left(\frac{e^{r_2 t_2}}{g_1 - r_2}\right)\left(x_1 - c_0 - c_1 t_1\right)}{\left|D\right|}$$

$$B = \frac{e^{r_2 t_1}\left(x_2 - c_0 - c_1 t_2\right) - e^{r_2 t_2}\left(x_1 - c_0 - c_1 t_1\right)}{\left(g_1 - r_2\right)\left|D\right|}$$

Determination of $\lambda(t)$ **via** $\frac{\partial H}{\partial u} = 0$ **and** $u(t)$

$$\left[\frac{\partial H}{\partial u} = e^{-rt}\left(k_1 + k_2 t + 2k_3 u\right) - \lambda = 0\right] \rightarrow \left[\lambda = e^{-rt}\left(k_1 + k_2 t + 2k_3 u\right)\right]$$

$$u(t) = Ae^{r_2 t} + z_1 + z_2 t$$

$$\lambda(t) = e^{-rt}\left(k_1 + k_2 t + 2k_3\left(Ae^{r_2 t} + z_1 + z_2 t\right)\right)$$

$$\lambda(t) = e^{-rt}\left(\left(k_1 + 2k_3 z_1\right) + \left(k_2 + 2k_3 z_2\right)t + 2k_3 Ae^{r_2 t}\right) \quad ; \quad r_2 = r - g_1$$

Conclusion:

The optimal adjoint variable path equation is:

$$\lambda(t) = e^{-rt}\left(\left(k_1 + 2k_3 z_1\right) + \left(k_2 + 2k_3 z_2\right)t\right) + 2k_3 A e^{-g_1 t}$$

Determination of the optimal objective function value:

$$J = \int_{t_1}^{t_2} e^{-rt} \left[\left(f_1 + f_2 t \right) x(t) + \left(k_1 + k_2 t \right) u(t) + k_3 \left(u(t) \right)^2 \right] dt$$

$$J = \int_{t_1}^{t_2} e^{-rt} \left[\left(f_1 + f_2 t \right) x(t) + \left(k_1 + k_2 t \right) u(t) + k_3 \left(u(t) \right)^2 \right] dt$$

$$J = \int_{t_1}^{t_2} K(t)\, dt$$

$$K = e^{-rt} \left[\left(f_1 + f_2 t \right) x(t) + \left(k_1 + k_2 t \right) u(t) + k_3 \left(u(t) \right)^2 \right]$$

$$x(t) = \frac{A}{g_1 - r_2} e^{r_2 t} + B e^{g_1 t} + c_0 + c_1 t$$

$$x(t) = c_0 + c_1 t + B e^{g_1 t} + D e^{r_2 t} \quad ; \quad D = \frac{A}{g_1 - r_2}$$

$$u(t) = z_1 + z_2 t + A e^{r_2 t}$$

$$K = e^{-rt}\left[\left(f_1 + f_2 t\right)x(t) + \left(k_1 + k_2 t\right)u(t) + k_3\left(u(t)\right)^2\right]$$

$$K = e^{-rt}\left[\left(f_1 + f_2 t\right)\left(c_0 + c_1 t + Be^{g_1 t} + De^{r_2 t}\right) + \left(k_1 + k_2 t\right)\left(z_1 + z_2 t + Ae^{r_2 t}\right) + k_3\left(z_1 + z_2 t + Ae^{r_2 t}\right)^2\right]$$

$$
\begin{aligned}
Ke^{rt} = {} & f_1 c_0 + f_1 c_1 t + f_1 B e^{g_1 t} + f_1 D e^{r_2 t} + f_2 c_0 t + f_2 c_1 t^2 + f_2 B t e^{g_1 t} + f_2 D t e^{r_2 t} \\
& + k_1 z_1 + k_1 z_2 t + k_1 A e^{r_2 t} + k_2 z_1 t + k_2 z_2 t^2 + k_2 A t e^{r_2 t} \\
& + k_3 (z_1)^2 + k_3 z_1 z_2 t + k_3 z_1 A e^{r_2 t} \\
& + k_3 z_1 z_2 t + k_3 (z_2)^2 t^2 + k_3 z_2 A t e^{r_2 t} \\
& + k_3 z_1 A e^{r_2 t} + k_3 z_2 A t e^{r_2 t} + k_3 A^2 e^{2 r_2 t}
\end{aligned}
$$

$$
\begin{aligned}
Ke^{rt} = \quad & \left(f_1c_0 + k_1z_1 + k_3(z_1)^2\right) + \left(f_1c_1 + f_2c_0 + k_1z_2 + k_2z_1 + 2k_3z_1z_2\right)t \\
& + \left(f_2c_1 + k_2z_2 + k_3(z_2)^2\right)t^2 + \left(f_1B\right)e^{g_1t} + \left(f_1D + k_1A + 2k_3z_1A\right)e^{r_2t} \\
& + \left(k_3A^2\right)e^{2r_2t} + \left(f_2B\right)te^{g_1t} + \left(f_2D + k_2A + 2k_3z_2A\right)te^{r_2t}
\end{aligned}
$$

$$
\begin{aligned}
Ke^{rt} = \quad & w_0 + w_1 t \\
& + w_2 t^2 + w_3 e^{g_1 t} + w_4 e^{r_2 t} \\
& + w_5 e^{2 r_2 t} + w_6 t e^{g_1 t} + w_7 t e^{r_2 t}
\end{aligned}
$$

$$K = w_0 e^{-rt} + w_1 t e^{-rt} + w_2 t^2 e^{-rt} + w_3 e^{(g_1 - r)t} + w_4 e^{(r_2 - r)t} + w_5 e^{(2r_2 - r)t} + w_6 t e^{(g_1 - r)t} + w_7 t e^{(r_2 - r)t}$$

$$K = w_0 e^{s_0 t} + w_1 t e^{s_0 t} + w_2 t^2 e^{s_0 t} + w_3 e^{s_1 t} + w_4 e^{s_2 t} + w_5 e^{s_3 t} + w_6 t e^{s_1 t} + w_7 t e^{s_2 t}$$

Now, we determine $I(t)$, the antiderivative of K.

$\frac{\partial I}{\partial t} = K$ **. Below, we exclude the constant of integration.**

$$I = w_0\left(\frac{e^{s_0 t}}{s_0}\right) + w_1\left(e^{s_0 t}\left(\frac{t}{s_0} - \frac{1}{(s_0)^2}\right)\right)$$

$$+ w_2\left(e^{s_0 t}\left(\frac{\left((s_0)^2 t^2 - 2 s_0 t + 2\right)}{(s_0)^3}\right)\right) + w_3\left(\frac{e^{s_1 t}}{s_1}\right) + w_4\left(\frac{e^{s_2 t}}{s_2}\right)$$

$$+ w_5\left(\frac{e^{s_3 t}}{s_3}\right) + w_6\left(e^{s_1 t}\left(\frac{t}{s_1} - \frac{1}{(s_1)^2}\right)\right) + w_7\left(e^{s_2 t}\left(\frac{t}{s_2} - \frac{1}{(s_2)^2}\right)\right)$$

$$J = I(t_2) - I(t_1)$$

Code section for calculation of J (in JavaScript):

```
var r2 = r-g1;
var z1 = 1/(2*k3)*(f1/r2 + f2/r2/r2 - k1);
var z2 = 1/(2*k3)*(f2/r2 - k2);
var c0 = 1/g1*(z1+(z2-g2)/g1 - g0);
var c1 = (z2-g2)/g1;
```

```
var sysdet = Math.exp(r2*t1)/(g1-r2)
*Math.exp(g1*t2) - Math.exp(r2*t2)
/(g1-r2)*Math.exp(g1*t1);

var A = ( (x1-c0-c1*t1)*Math.exp(g1*t2)
-(x2-c0-c1*t2)*Math.exp(g1*t1) )
-/sysdet;

var B = ( Math.exp(r2*t1)/(g1-r2)*(x2-c0-c1*t2)
– Math.exp(r2*t2)/(g1-r2)*(x1-c0-c1*t1))
/sysdet;
```

```
var D = A/(g1-r2);
var w0 = f1*c0 + k1*z1 + k3*z1*z1;
var w1 = f1*c1 + f2*c0 + k1*z2 + k2*z1 + 2*k3*z1*z2;
var w2 = f2*c1 + k2*z2 + k3*z2*z2;
var w3 = f1*B;
var w4 = f1*D + k1*A + 2*k3*z1*A;
var w5 = k3*A*A;
var w6 = f2*B;
var w7 = f2*D + k2*A + 2*k3*z2*A;
```

```
var s0 = - r;
var s1 = g1 - r;
var s2 = r2 - r;
var s3 = 2*r2 - r;
```

```
var AntidK2 =
    w0*(1/s0*Math.exp(s0*t2))
  + w1*(Math.exp(s0*t2)*(t2/s0 - 1/(s0*s0)))
  + w2*(Math.exp(s0*t2)*((s0*s0*t2*t2-2*s0*t2+2)
          /(s0*s0*s0)))
  + w3*(1/s1*Math.exp(s1*t2))
  + w4*(1/s2*Math.exp(s2*t2))
  + w5*(1/s3*Math.exp(s3*t2))
  + w6*(Math.exp(s1*t2)*(t2/s1 - 1/(s1*s1)))
  + w7*(Math.exp(s2*t2)*(t2/s2 - 1/(s2*s2))) ;
```

```
var AntidK1 =
        w0*(1/s0*Math.exp(s0*t1))
      + w1*(Math.exp(s0*t1)*(t1/s0 - 1/(s0*s0)))
      + w2*(Math.exp(s0*t1)*((s0*s0*t1*t1-2*s0*t1 + 2)
            /(s0*s0*s0)))
      + w3*(1/s1*Math.exp(s1*t1))
      + w4*(1/s2*Math.exp(s2*t1))
      + w5*(1/s3*Math.exp(s3*t1))
      + w6*(Math.exp(s1*t1)*(t1/s1 - 1/(s1*s1)))
      + w7*(Math.exp(s2*t1)*(t1/s2 - 1/(s2*s2))) ;

var Integral = AntidK2 - AntidK1;

var J = Integral;
```

General observations of optimal solutions:

First we consider the following objective function. Note that the stock level does not influence the objective function directly in this first version of the problem.

$$J = \int_{t1}^{t2} e^{-rt} R(u(t))dt$$

$$\dot{x} = F(x) - u$$

The Hamiltonian function is:

$$H = e^{-rt} R(u) + \lambda(F(x) - u)$$

$$\frac{\partial H}{\partial u} = e^{-rt} R_u - \lambda$$

When we have an interior optimum, the first order derivative of the objective function with respect to the control variable is 0. As a result, we find that the present value of the marginal profit,

$$e^{-rt} R_u(u(t))$$

is equal to the marginal resource value at the same point in time,

$\lambda(t)$ **.**

$$\left(\frac{\partial H}{\partial u} = 0 \right) \Rightarrow \left(e^{-rt} R_u = \lambda \right)$$

The adjoint equation says that

$$\frac{\partial H}{\partial x} = -\frac{\partial \lambda}{\partial t}$$

Since $\frac{\partial H}{\partial x} = F_x \lambda$ **, we know that** $F_x \lambda = -\dot{\lambda}$

Now, we make the following definition:

$$F(x) = \int_0^x G(z)dz$$

The resource, with total stock

x **, consists of many different parts.**

Different parts of the resource have different relative growth. A particular part of the resource,

z **, has relative growth** $G(z)$ **.**

We may define the relative growth of that particular unit as

$$\frac{\dot{y}(z)}{y(z)}$$

($y(z)$ **is the stock level of that particular unit and**

$\dot{y}$ **is the growth.)**

When we integrate,

$$F(x) = \int_0^x G(z)\,dz$$

we arrange the different particular resource units in such a way that

$$F_{xx}(x) \leq 0 \quad , \forall x$$

$$G(z) = \frac{\dot{y}(z)}{y(z)}$$

$$F_x(x) = G(x) = \frac{\dot{y}(x)}{y(x)}$$

$$\frac{\dot{y}(x)}{y(x)}\lambda = -\dot{\lambda}$$

$$\frac{\dot{\lambda}}{\lambda} = -\frac{\dot{y}(x)}{y(x)}$$

Observation:

The relative decrease of the marginal resource value over time is equal to the relative growth of the marginal resource unit.

$$\frac{\dot{\lambda}}{\lambda} = -\frac{\dot{y}(x)}{y(x)}$$

Example with explanation from forestry:

If the marginal cubic metre grows by 5% per year (and will give 0.05 new cubic metres next year) then the value of the marginal cubic metre this year is 5% higher than the value of the marginal cubic metre next year.

The value of an already existing cubic metre must be higher than the value of a cubic metre in the future year since the cubic metre that we already have may give us more than one cubic metre in the future year, thanks to the growth.

The relative marginal resource value decrease is higher

in case the relative marginal growth

$\frac{\dot{y}(x)}{y(x)}$ **is higher.**

$$\frac{\dot{\lambda}}{\lambda} = -\frac{\dot{y}(x)}{y(x)}$$

The relationship between the speed of change of the relative marginal resource value and direct valuation of the stock level

Here, we add

$$\int_{t1}^{t2} e^{-rt} W(x) dt$$

to the objective function.

As a result, we get:

$$J = \int_{t1}^{t2} e^{-rt}\left(R(u(t)) + W(x)\right)dt$$

$$\dot{x} = F(x) - u$$

The Hamiltonian function becomes:

$$H = e^{-rt}\left(R(u) + W(x)\right) + \lambda\left(F(x) - u\right)$$

These derivatives are obtained:

$$\frac{\partial H}{\partial u} = e^{-rt} R_u - \lambda$$

$$\frac{\partial H}{\partial x} = e^{-rt} W_x + F_x \lambda$$

The adjoint equation gives a modified result:

$$\left(\frac{\partial H}{\partial x} = -\frac{\partial \lambda}{\partial t}\right) \Rightarrow$$

$$e^{-rt}W_x + F_x\lambda = -\dot{\lambda}$$

$$\frac{e^{-rt}W_x}{\lambda} + F_x = -\frac{\dot{\lambda}}{\lambda}$$

$$\frac{e^{-rt}W_x}{\lambda} + \frac{\dot{y}}{y} = -\frac{\dot{\lambda}}{\lambda}$$

$$\frac{\dot{\lambda}}{\lambda} = -\frac{\dot{y}}{y} - \frac{e^{-rt}W_x}{\lambda}$$

Observation:

$\lambda > 0$ **implies that**

$\frac{\dot{\lambda}}{\lambda}$ **becomes more negative if** W_x **increases.**

Explanation and connection to forestry:

$$W_x > 0$$

means that the stock gives an “instant contribution” to the objective function.

One such case is if the forest owner continuously gets paid for the amount of stored CO_2 in the forest. In such a case, one cubic metre that exists now (and that will not be instantly harvested),

will give contributions in this period and in the following period.

One cubic metre that will exist in the next period, but not already in this period, will not “be able” to give the contribution during this period. As a result, the fact that we get “instant contributions”, directly from the stock, means that the relative marginal value of the stock falls faster over time.

Software for the optimal control model

http://www.lohmander.com/CM/CM.htm

CM

Peter Lohmander

080821

Parameter Table (yellow)

t1	t2	r
f1	f2	
k1	k2	k3
g0	g1	g2
x1	x2	

Results (red)

Click here to derive the results!

z1	z2	r2
c0	c1	c2
A	B	J
x0	u0	lambda0
x5	u5	lambda5
x10	u10	lambda10
x15	u15	lambda15
x20	u20	lambda20
x25	u25	lambda25
x30	u30	lambda30

$$\max\left\{ J = \int_{t_1}^{t_2} e^{-rt}\left(\left(f_1 + f_2 t\right)x + \left(k_1 + k_2 t\right)u + k_3 u^2\right)dt \right\}$$

$$\dot{x} = g_0 + g_1 x + g_2 t - u$$

$$x(t_1) = x_1; \quad x(t_2) = x_2$$

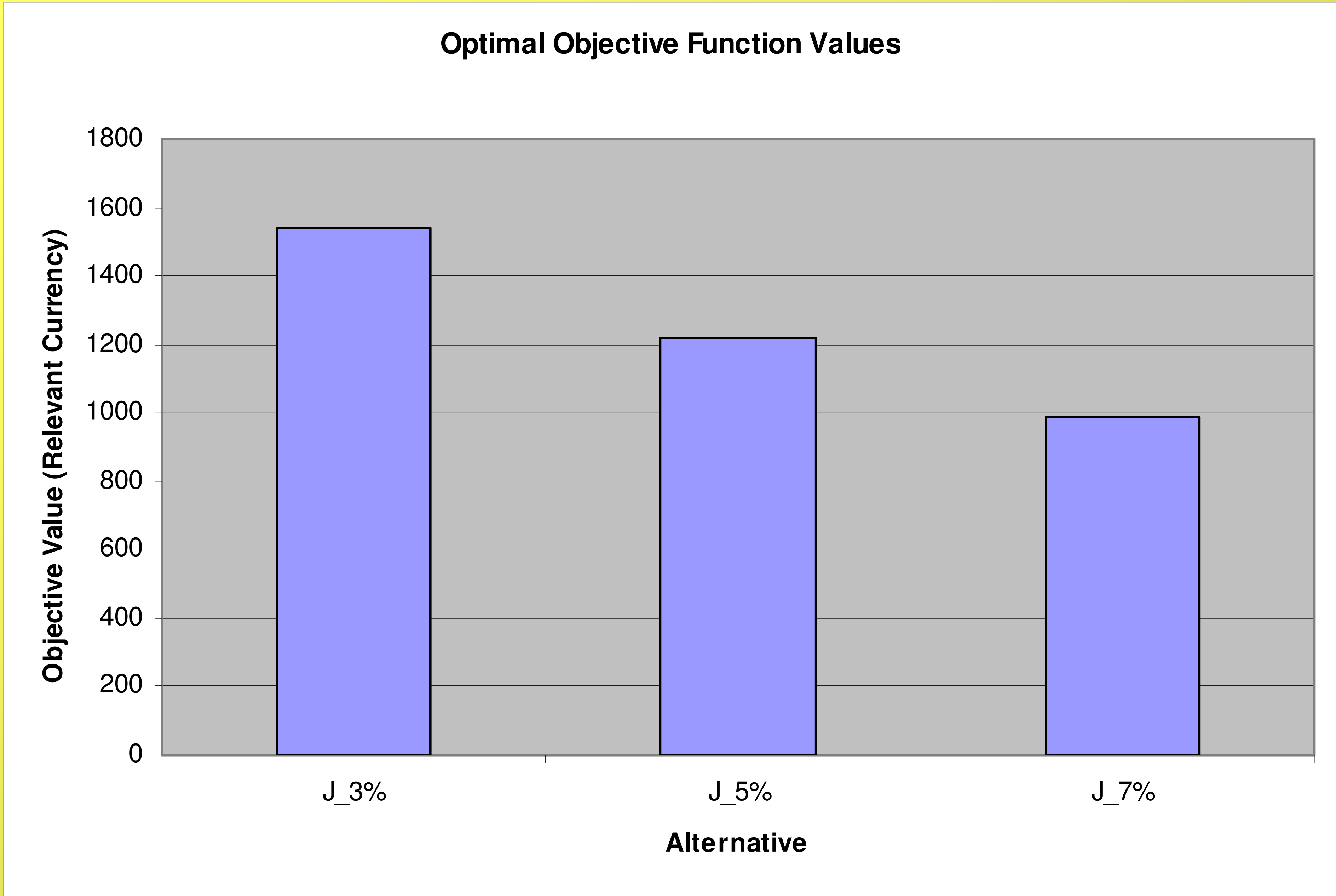
t1	t2	r
0	30	.05
f1	f2	
0	0	
k1	k2	k3
1000	0	-3
g0	g1	g2
75	.01	0
x1	x2	
3100	2500	

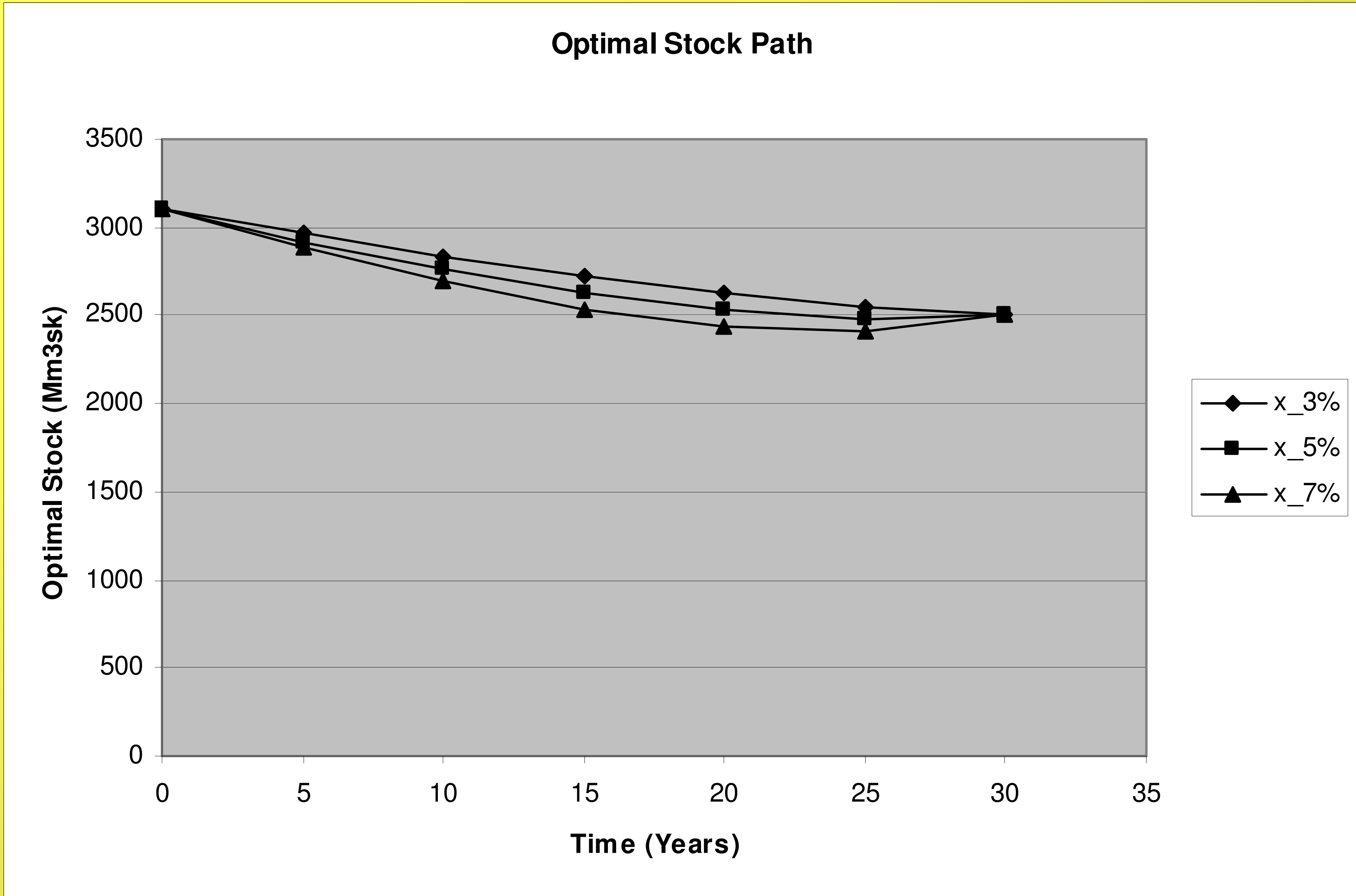
J	1216344		

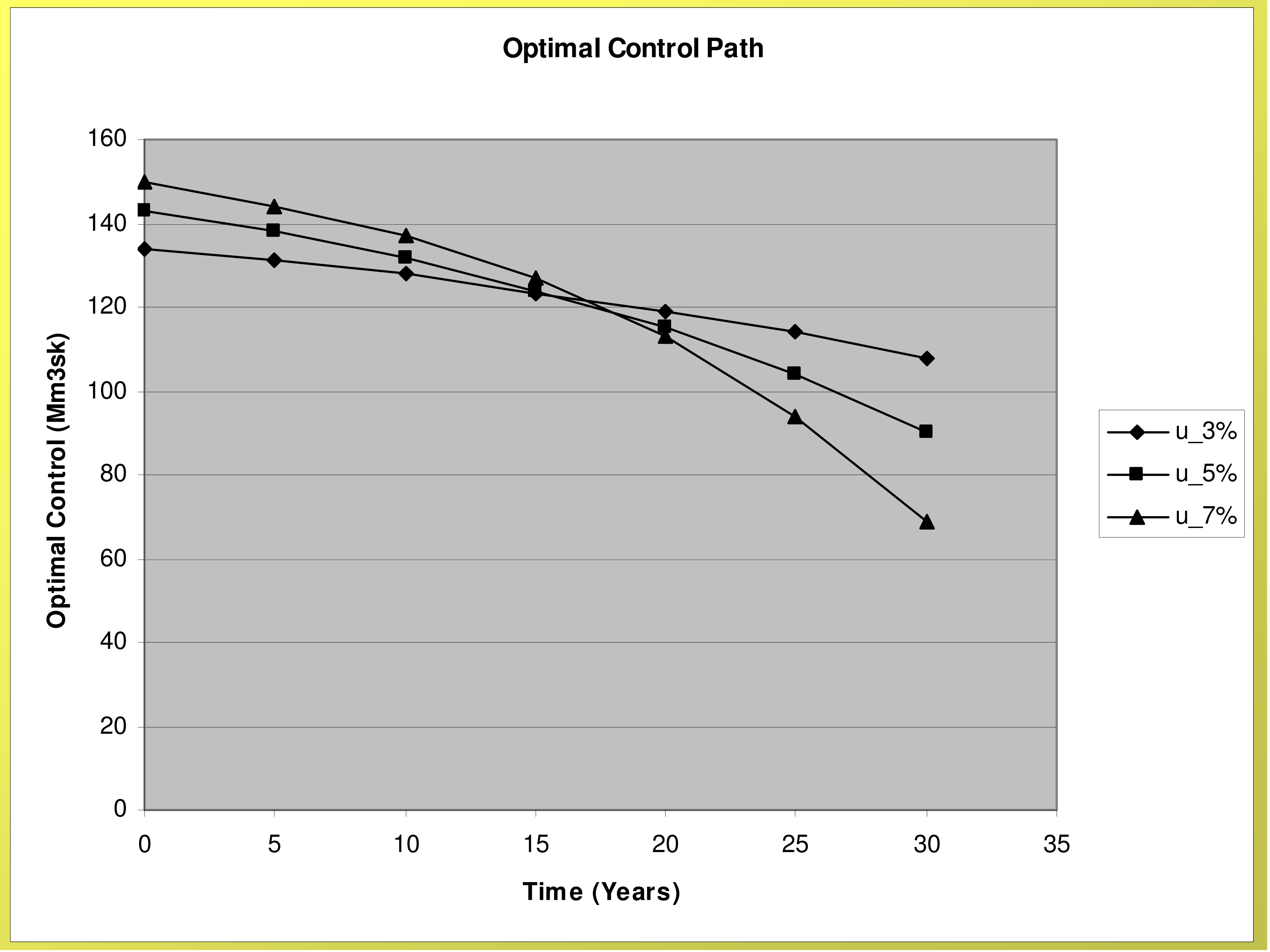
t	x	u	Lambda
0	3100	143	139
5	2920	138	132
10	2761	132	126
15	2628	124	120
20	2533	115	114
25	2485	104	108
30	2500	90	103

Optimal strategy and dynamic effects of

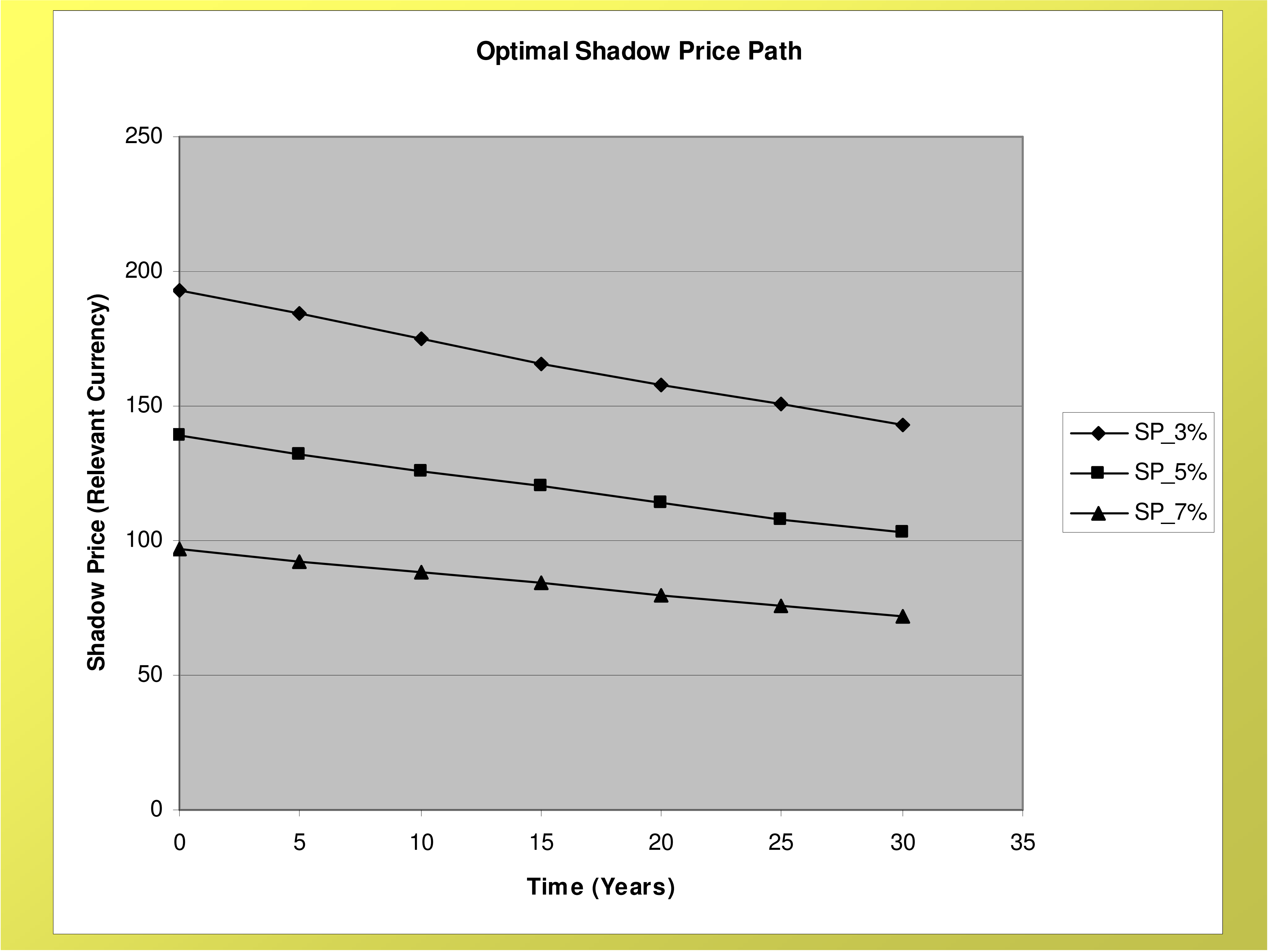
the rate of interest


Optimal Objective Function Values
Objective Value (Relevant Currency)
1800
1600
1400
1200
1000
800
600
400
200
0
J_3%
J_5%
J_7%
Alternative


Optimal Stock Path
Optimal Stock (Mm3sk)
3500
3000
2500
2000
1500
1000
500
0
0
5
10
15
20
25
30
35
Time (Years)
x_3%
x_5%
x_7%



Optimal Control Path
Optimal Control (Mm3sk)
Time (Years)
160
140
120
100
80
60
40
20
0
0
5
10
15
20
25
30
35
u_3%
u_5%
u_7%



Optimal Shadow Price Path
Shadow Price (Relevant Currency)
250
200
150
100
50
0
0
5
10
15
20
25
30
35
Time (Years)
SP_3%
SP_5%
SP_7%

Comparisions with alternatives that are not optimal:

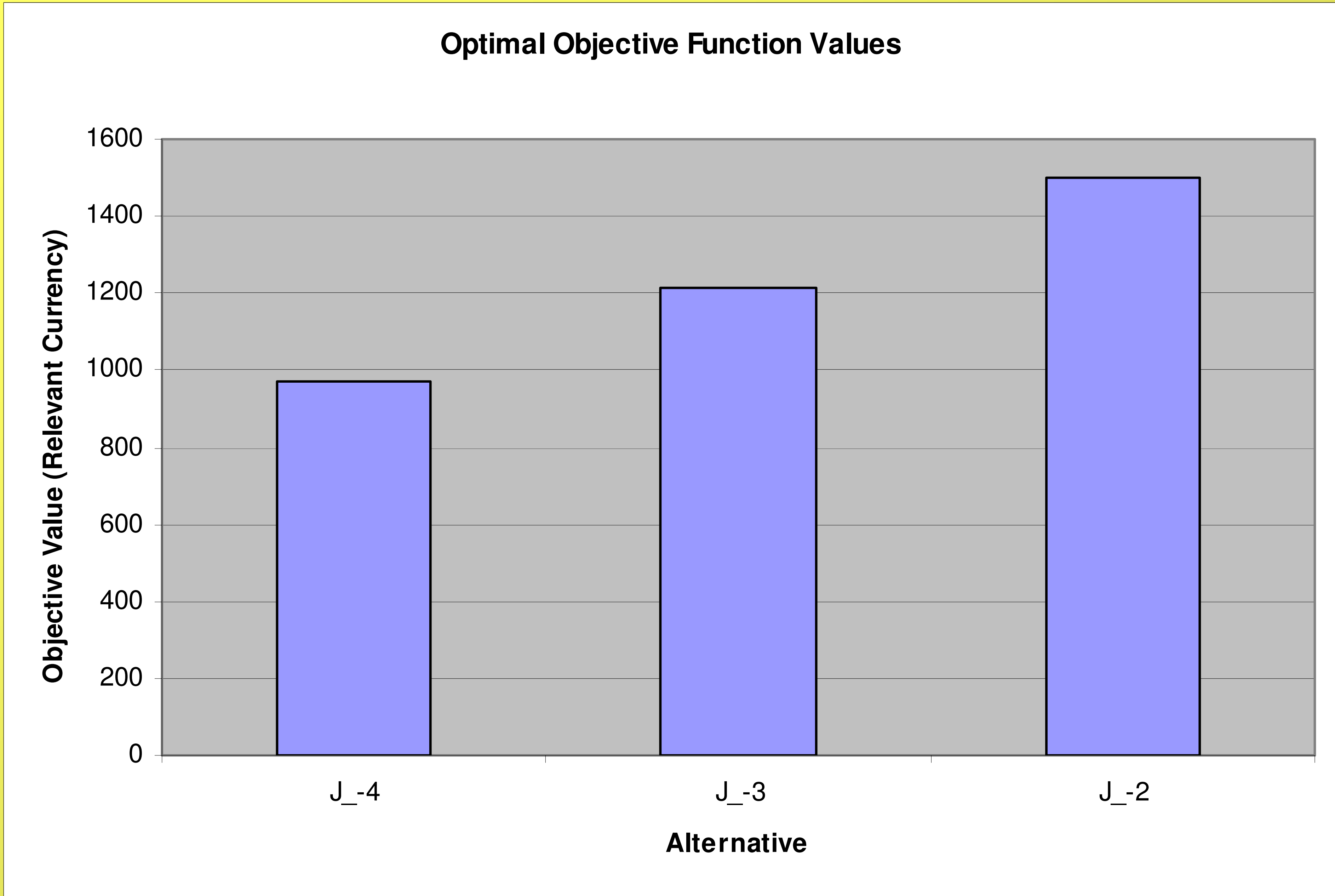
$$N_1 = \int_0^{30} e^{-.05t}\left(1000 - 3*106\right)106\,dt \approx 1.123229489 \cdot 10^6$$

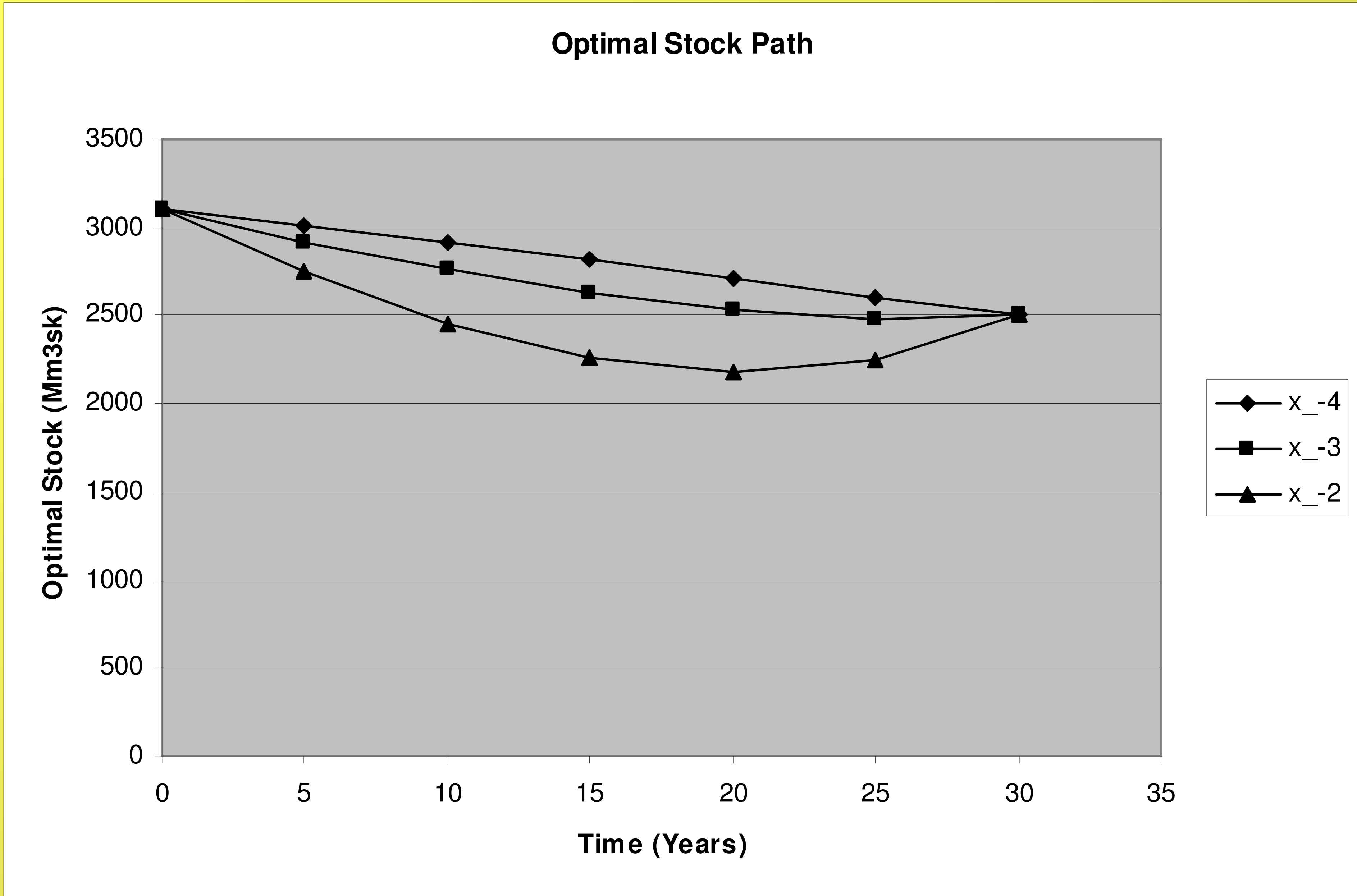
$$N_2 = \int_0^{30} e^{-.05t}\left(1000 - 3*86\right)86\,dt \approx 9.914723644 \cdot 10^5$$

Optimal strategy and dynamic effects of

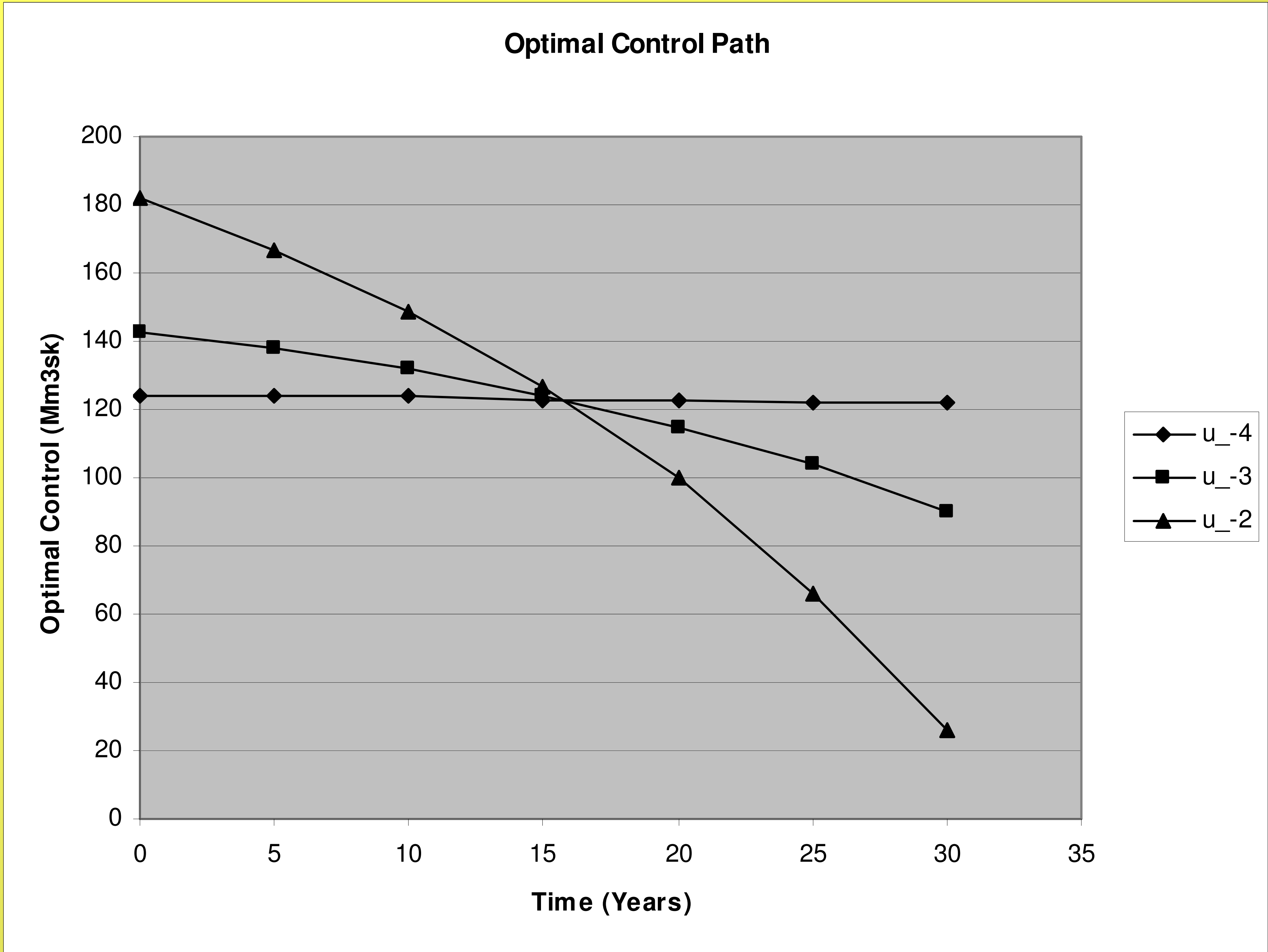
the slope of the demand function

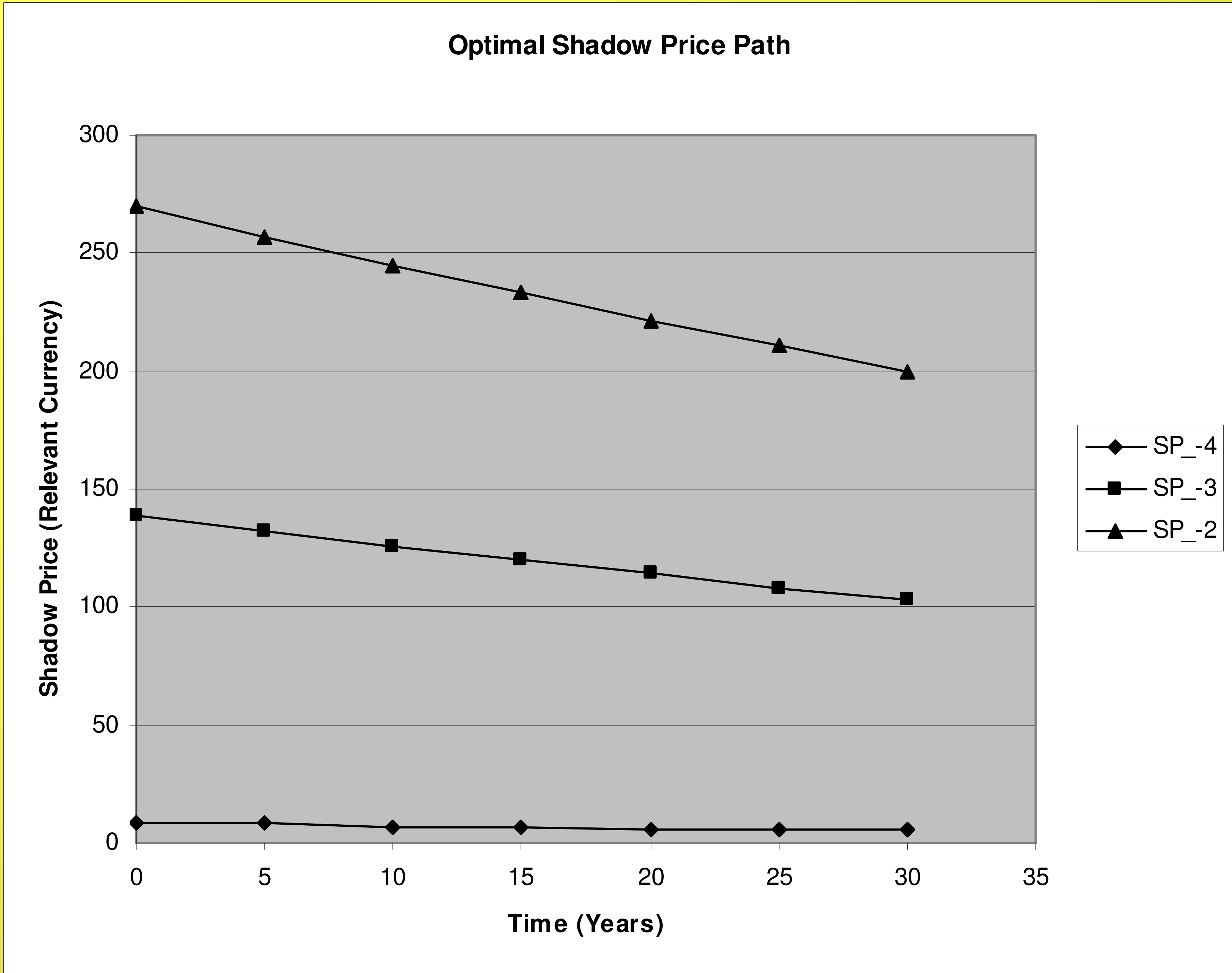
t1	t2	r
0	30	.05
f1	f2	
0	0	
k1	k2	k3
1000	0	-3
g0	g1	g2
75	.01	0
x1	x2	
3100	2500	


Optimal Objective Function Values
Objective Value (Relevant Currency)
1600
1400
1200
1000
800
600
400
200
0
J_-4
J_-3
J_-2
Alternative



Optimal Stock Path
Optimal Stock (Mm3sk)
Time (Years)
3500
3000
2500
2000
1500
1000
500
0
0
5
10
15
20
25
30
35
x_-4
x_-3
x_-2


Optimal Control Path
200
180
160
140
120
100
80
60
40
20
0
Optimal Control (Mm3sk)
0
5
10
15
20
25
30
35
Time (Years)
u_-4
u_-3
u_-2



Optimal Shadow Price Path
Shadow Price (Relevant Currency)
300
250
200
150
100
50
0
0
5
10
15
20
25
30
35
Time (Years)
SP_-4
SP_-3
SP_-2

Comparisions with two alternatives:

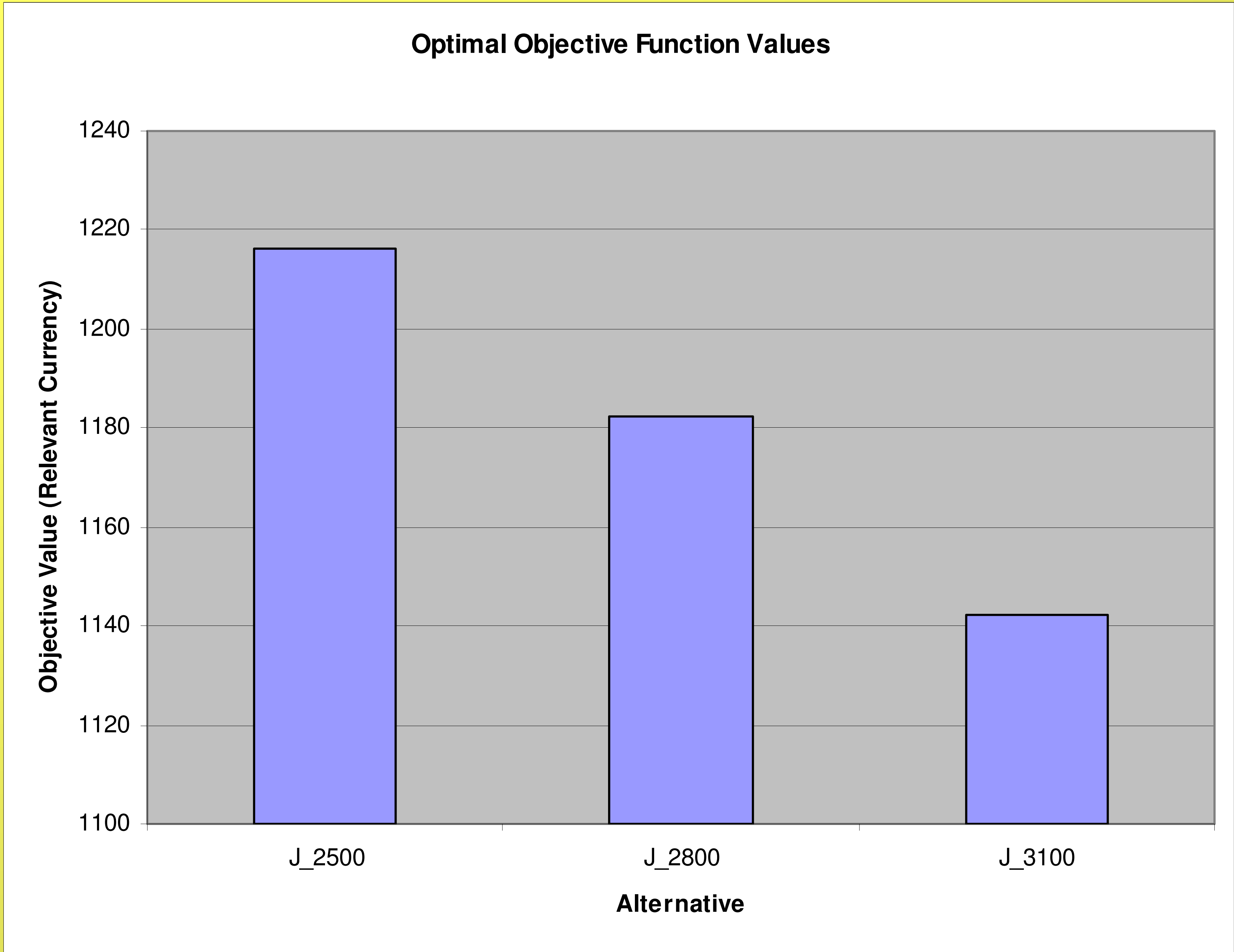
$$N_3 = \int_0^{30} e^{-.05t}\left(1000 - 2*106\right)106\, dt \approx 1.297807679 \cdot 10^6$$

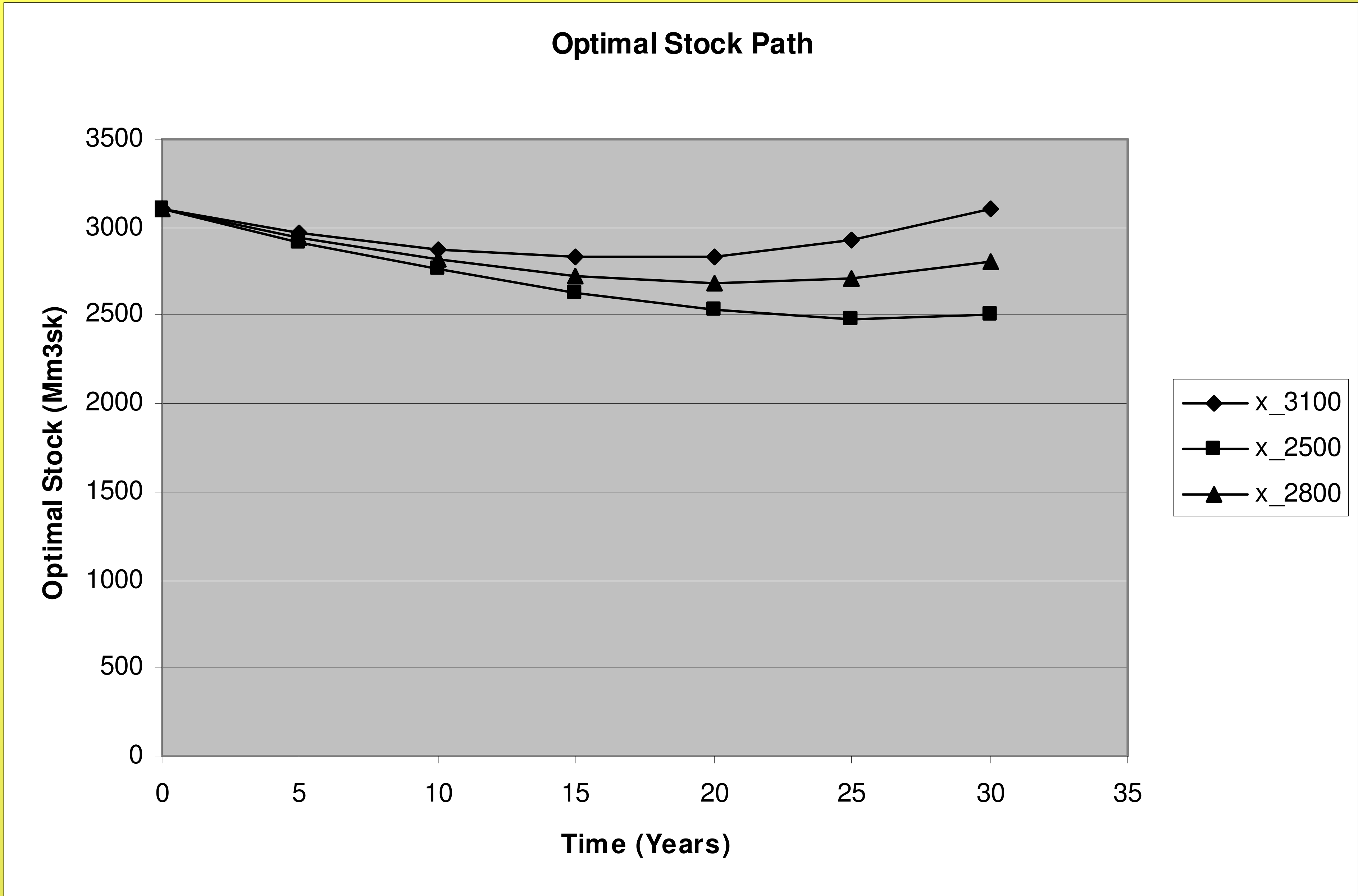
$$N_4 = \int_0^{30} e^{-.05t}\left(1000 - 2*(181 - 5*t)\right)(181 - 5*t)\, dt \approx 1.397224592 \cdot 10^6$$

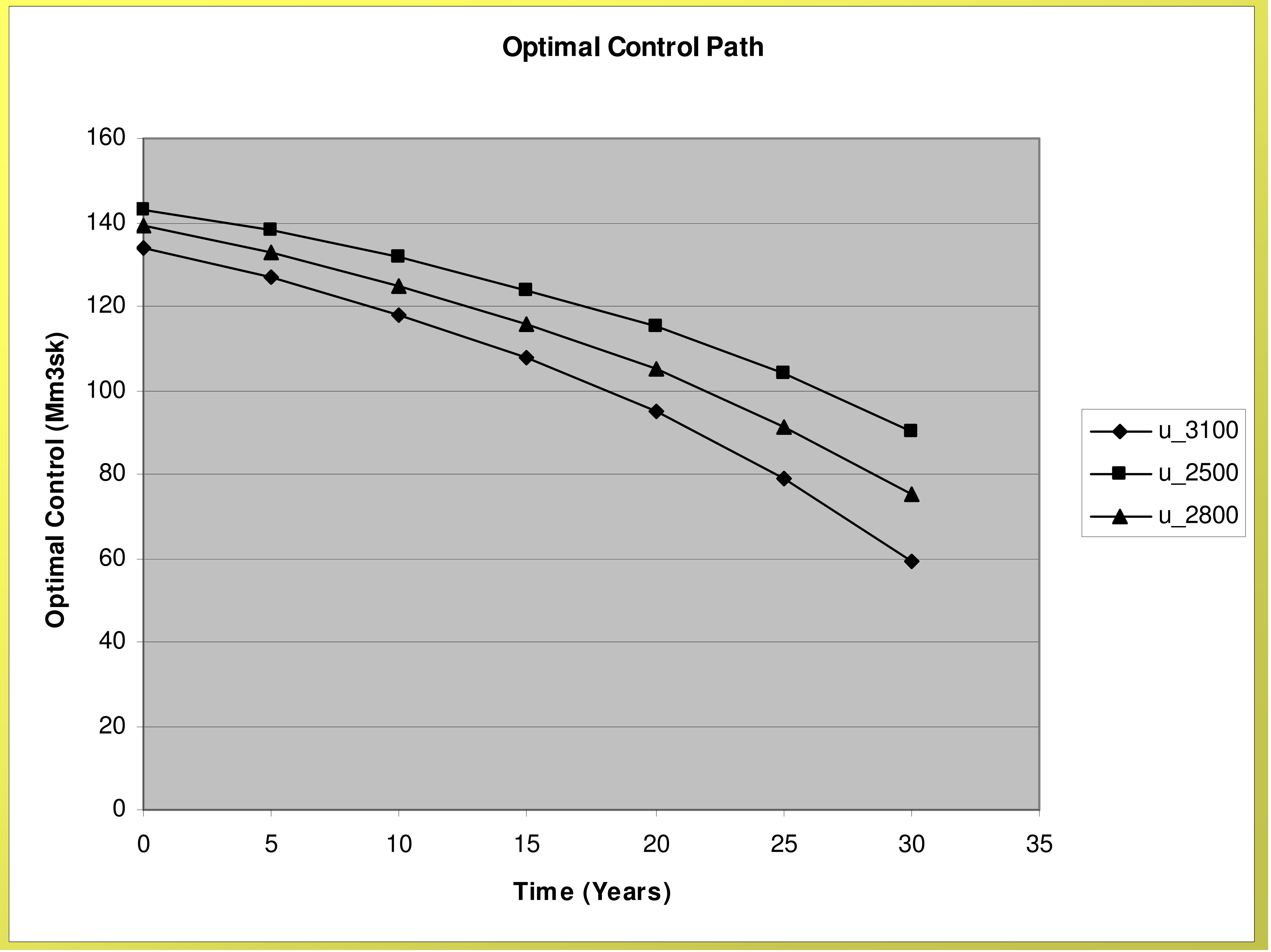
Optimal strategy and dynamic effects of

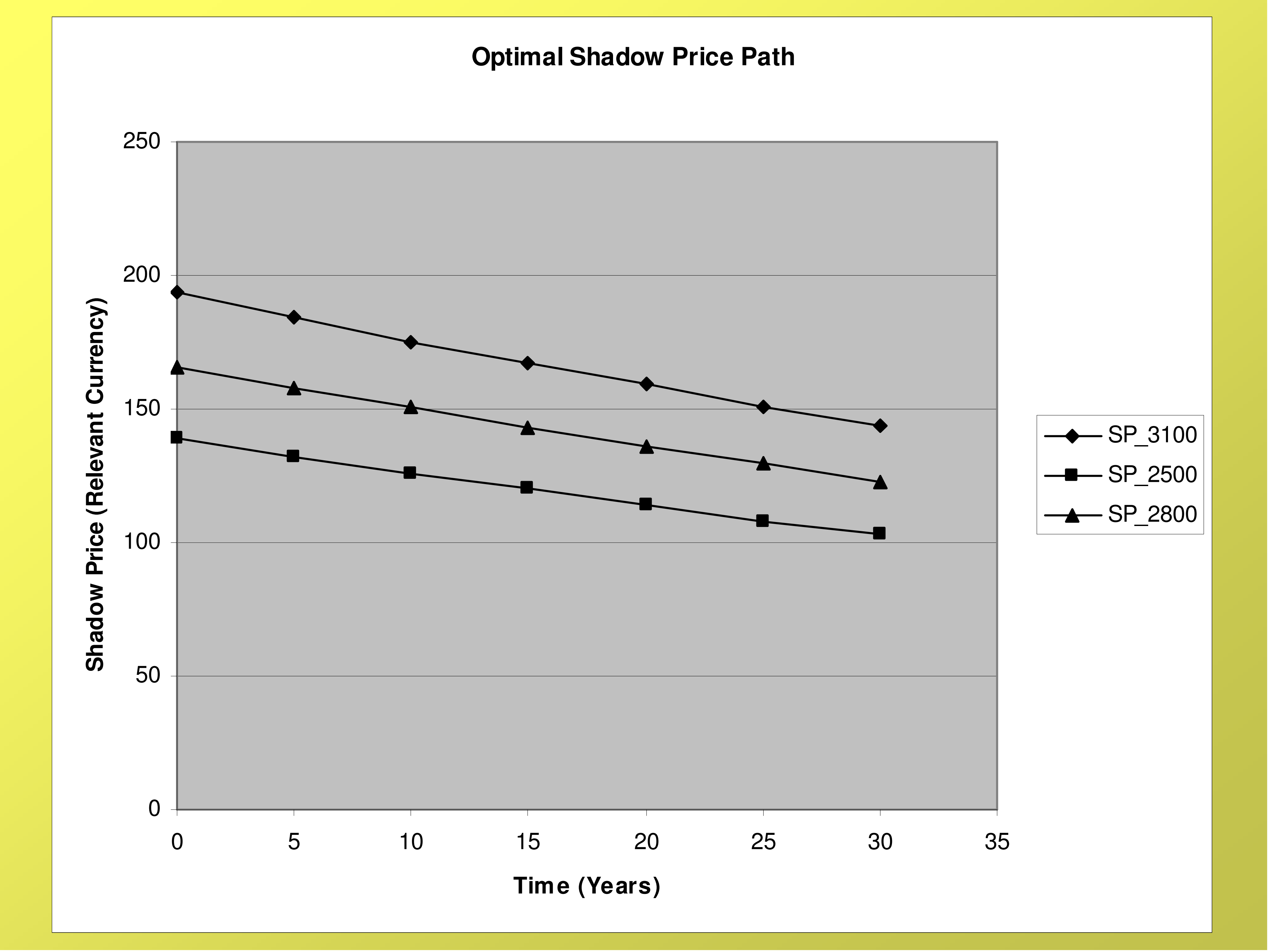
the terminal condition

t1	t2	r
0	30	.05
f1	f2	
0	0	
k1	k2	k3
1000	0	-3
g0	g1	g2
75	.01	0
x1	x2	
3100	2500	


Optimal Objective Function Values
Objective Value (Relevant Currency)
1240
1220
1200
1180
1160
1140
1120
1100
J_2500
J_2800
J_3100
Alternative


Optimal Stock Path
Optimal Stock (Mm3sk)
3500
3000
2500
2000
1500
1000
500
0
0
5
10
15
20
25
30
35
Time (Years)
x_3100
x_2500
x_2800

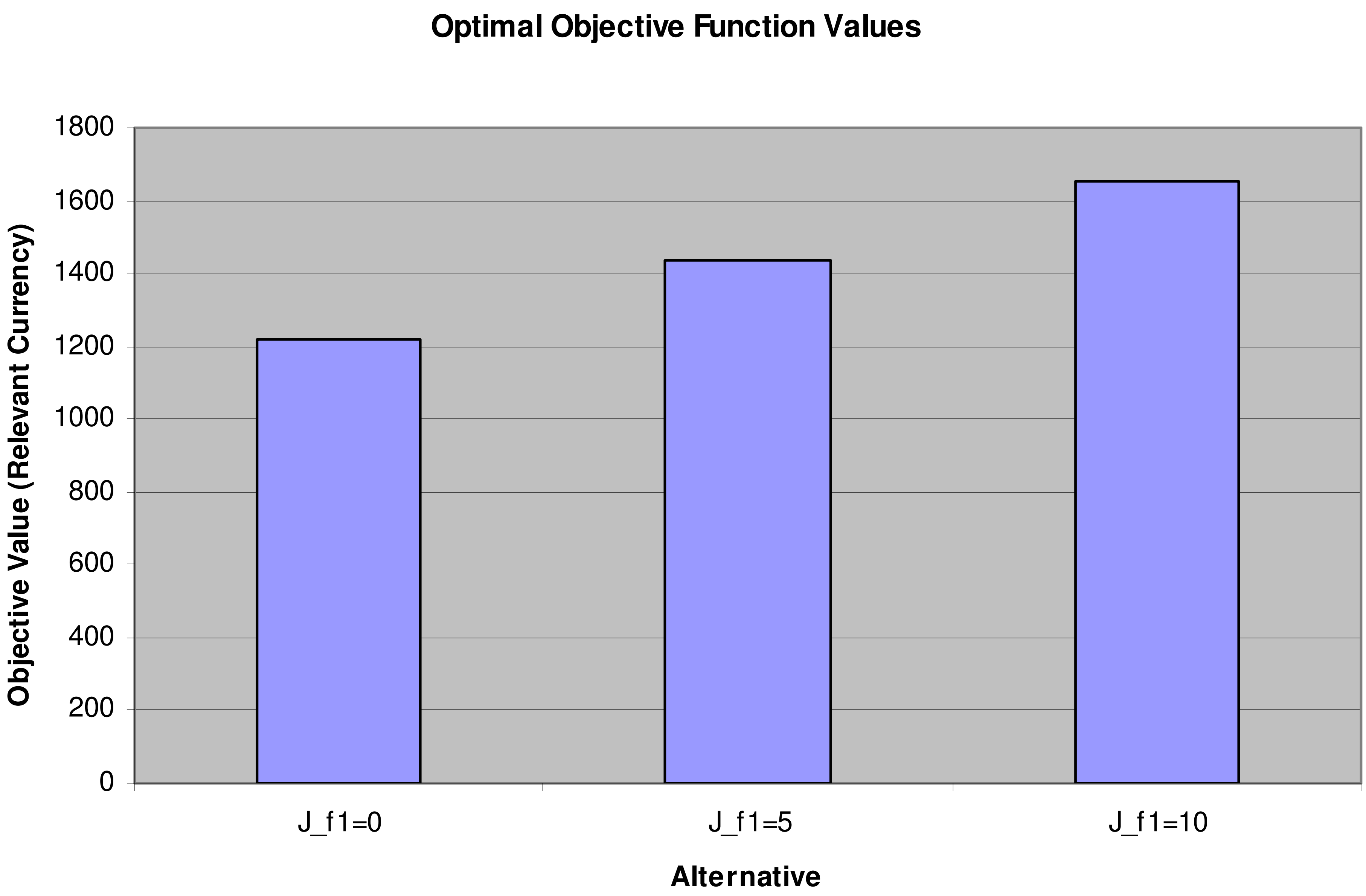

Optimal Control Path
Optimal Control (Mm3sk)
160
140
120
100
80
60
40
20
0
0
5
10
15
20
25
30
35
Time (Years)
u_3100
u_2500
u_2800

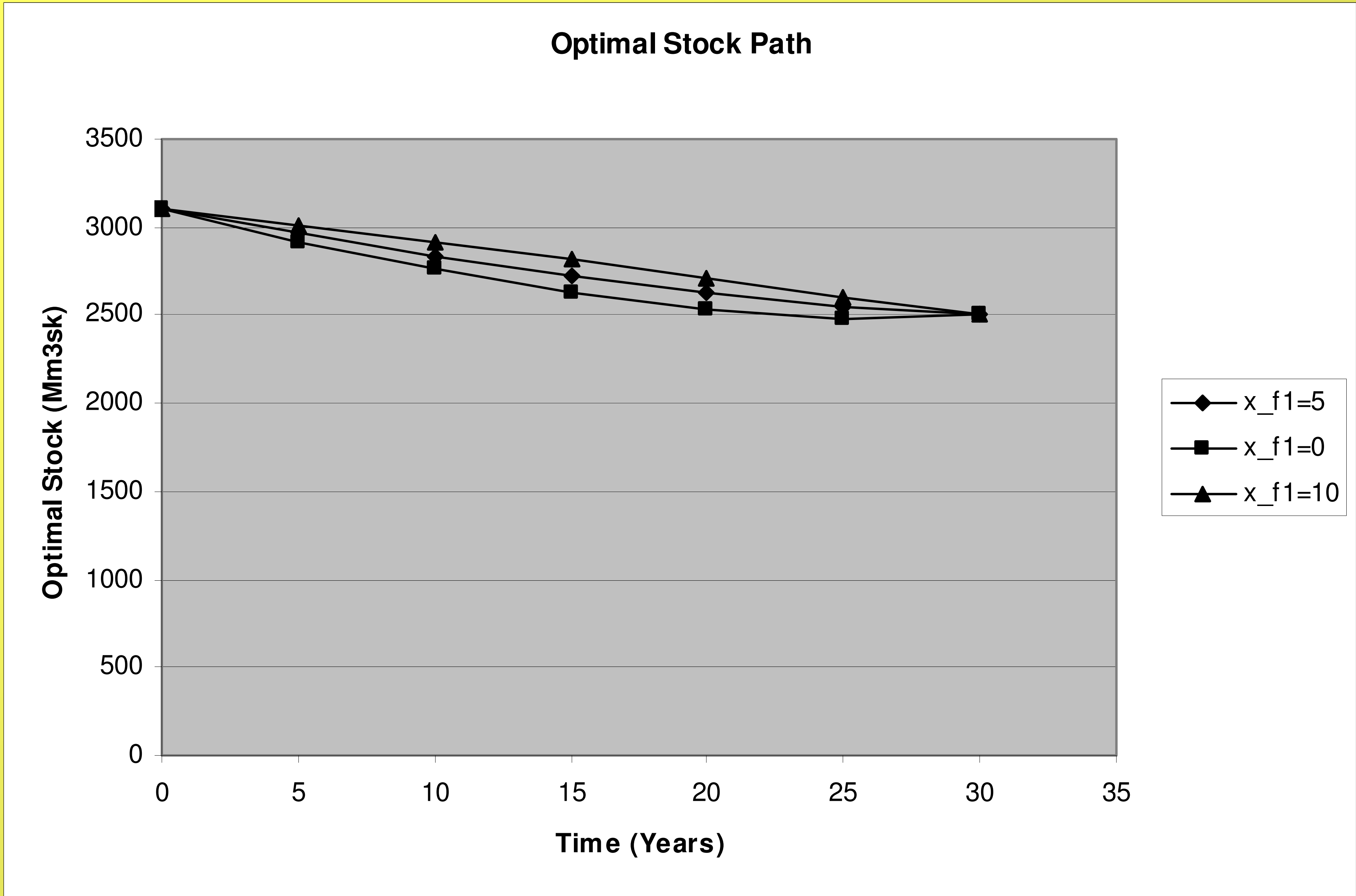

Optimal Shadow Price Path
250
200
150
100
50
0
Shadow Price (Relevant Currency)
0
5
10
15
20
25
30
35
Time (Years)
SP_3100
SP_2500
SP_2800

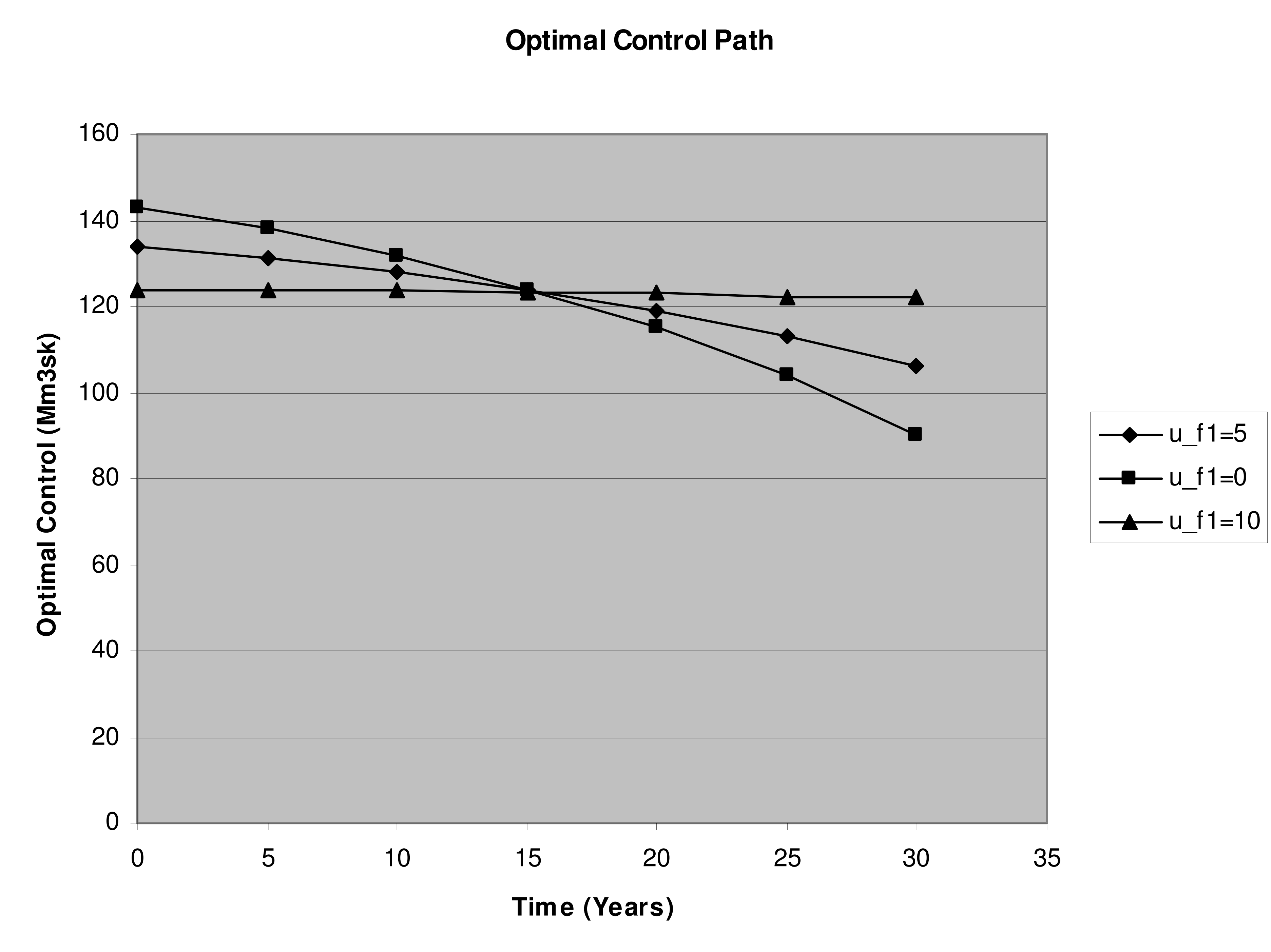
Optimal strategy and dynamic effects of

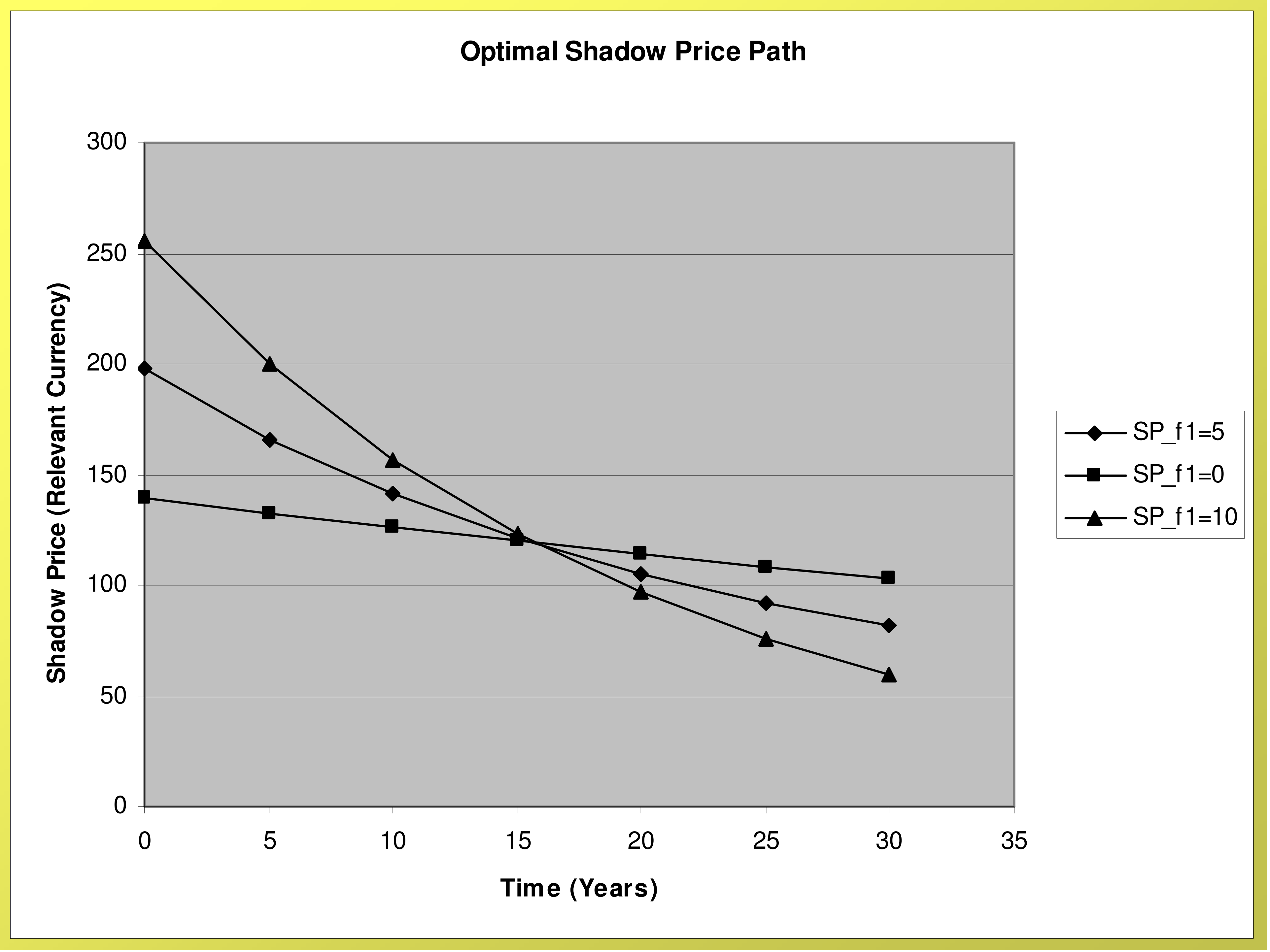
continuous stock level valuation

t1	**t2**	**r**
0	30	.05
f1	**f2**	
0	0	
k1	**k2**	**k3**
1000	0	-3
g0	**g1**	**g2**
75	.01	0
x1	**x2**	
3100	2500	


Optimal Objective Function Values
Objective Value (Relevant Currency)
1800
1600
1400
1200
1000
800
600
400
200
0
J_f1=0
J_f1=5
J_f1=10
Alternative


Optimal Stock Path
Optimal Stock (Mm3sk)
3500
3000
2500
2000
1500
1000
500
0
0
5
10
15
20
25
30
35
Time (Years)
x_f1=5
x_f1=0
x_f1=10


Optimal Control Path
Optimal Control (Mm3sk)
Time (Years)
0
20
40
60
80
100
120
140
160
0
5
10
15
20
25
30
35
u_f1=5
u_f1=0
u_f1=10

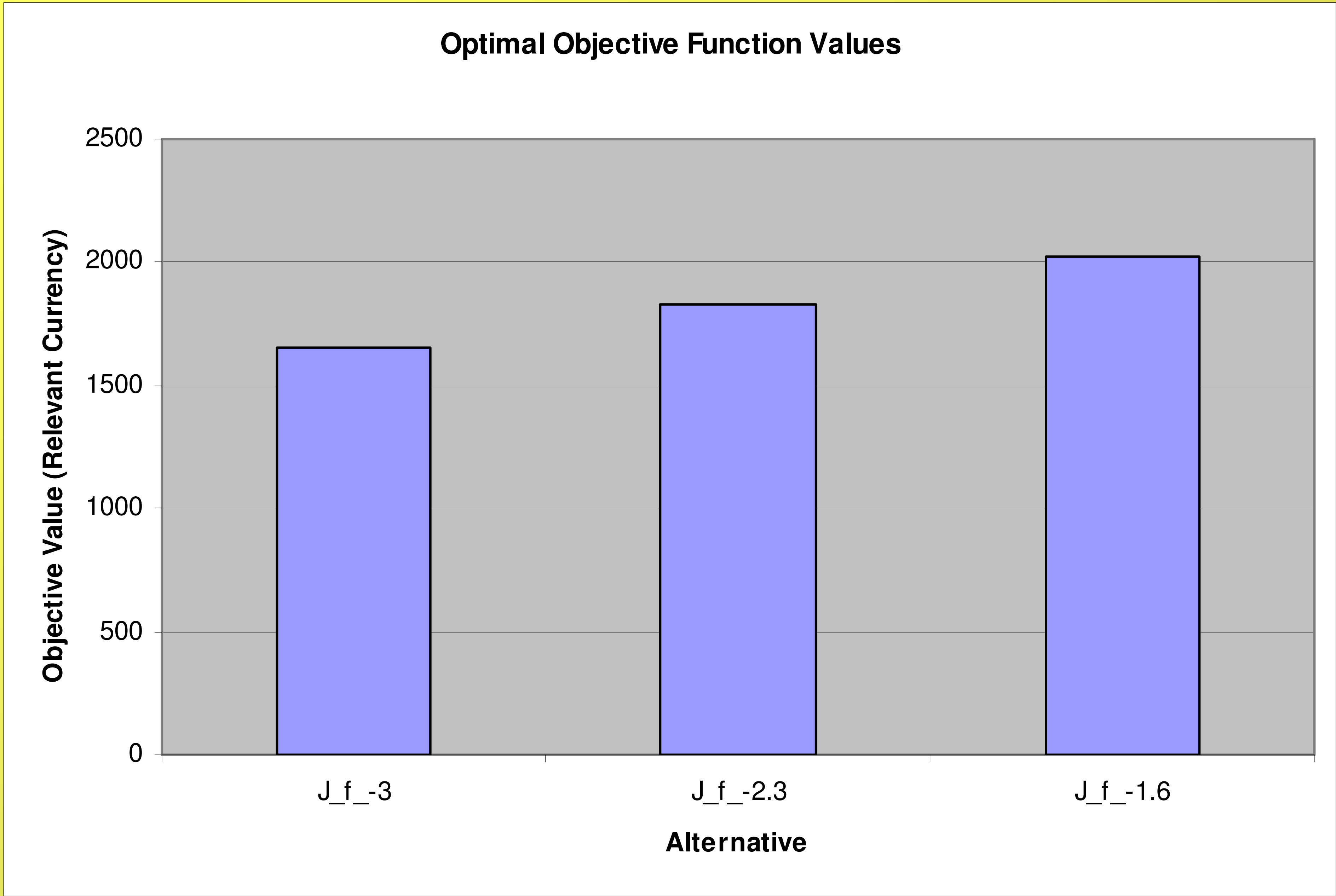


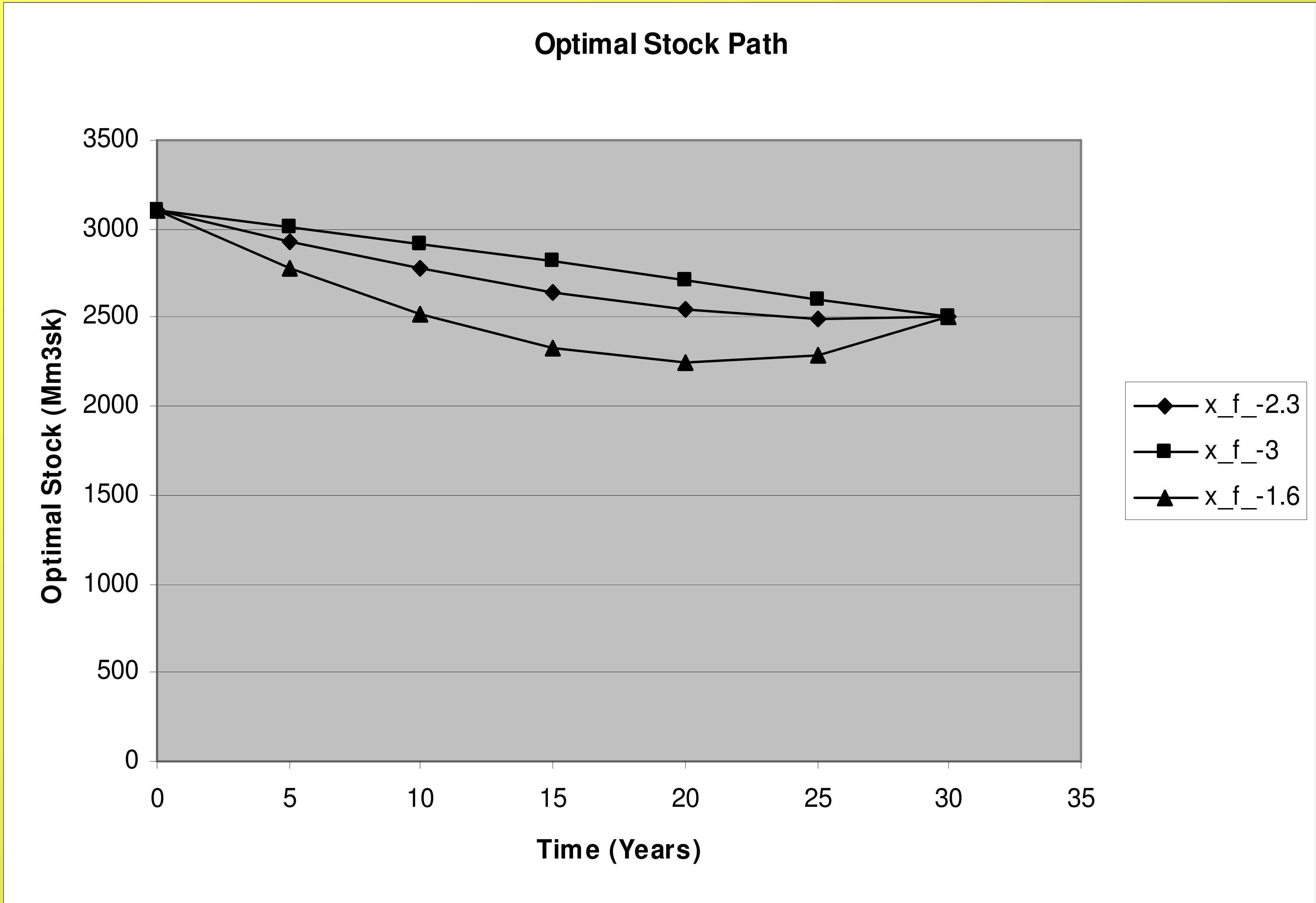
Optimal Shadow Price Path
300
250
200
150
100
50
0
0
5
10
15
20
25
30
35
Shadow Price (Relevant Currency)
Time (Years)
SP_f1=5
SP_f1=0
SP_f1=10

Optimal strategy and dynamic effects of

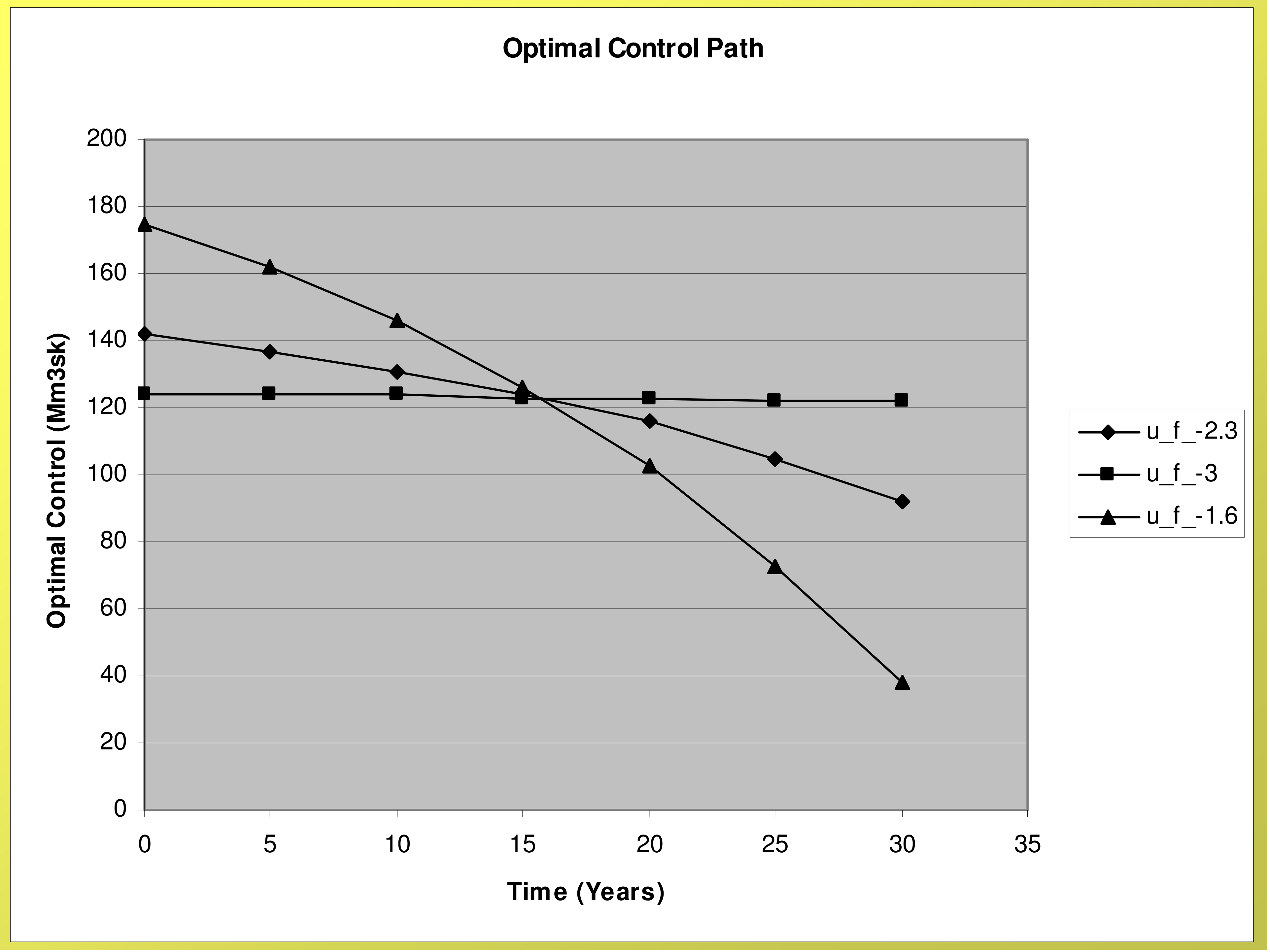
continuous stock level valuation in combination with variations of the the slope of the demand function

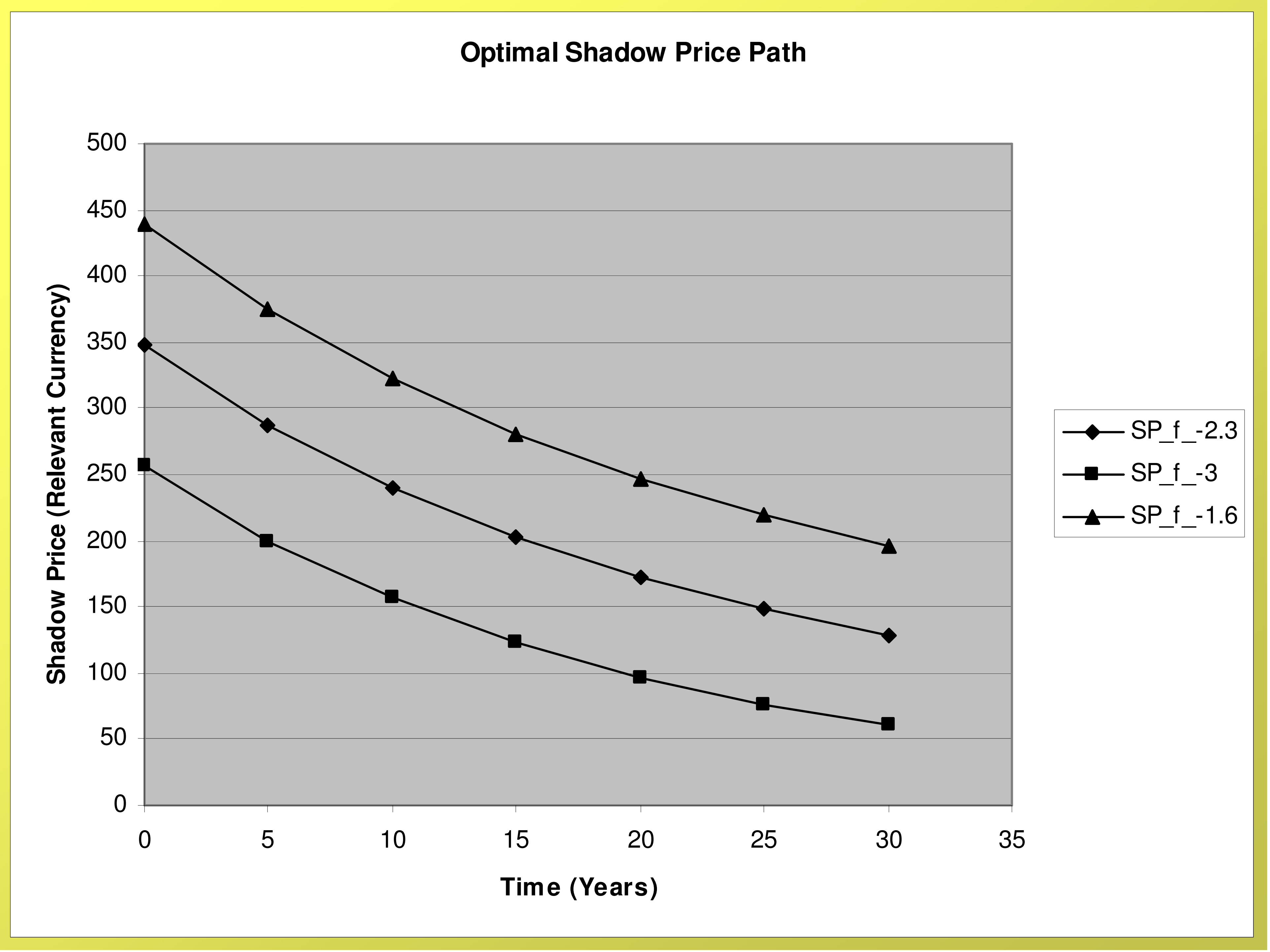
0	30	.05
10	0	
1000	0	-3
75	.01	0
3100	2500	


Optimal Objective Function Values
Objective Value (Relevant Currency)
2500
2000
1500
1000
500
0
J_f_-3
J_f_-2.3
J_f_-1.6
Alternative



Optimal Stock Path
Optimal Stock (Mm3sk)
3500
3000
2500
2000
1500
1000
500
0
0
5
10
15
20
25
30
35
Time (Years)
x_f_-2.3
x_f_-3
x_f_-1.6


Optimal Control Path
Optimal Control (Mm3sk)
200
180
160
140
120
100
80
60
40
20
0
0
5
10
15
20
25
30
35
Time (Years)
u_f_-2.3
u_f_-3
u_f_-1.6


Optimal Shadow Price Path
Shadow Price (Relevant Currency)
500
450
400
350
300
250
200
150
100
50
0
0
5
10
15
20
25
30
35
Time (Years)
SP_f_-2.3
SP_f_-3
SP_f_-1.6

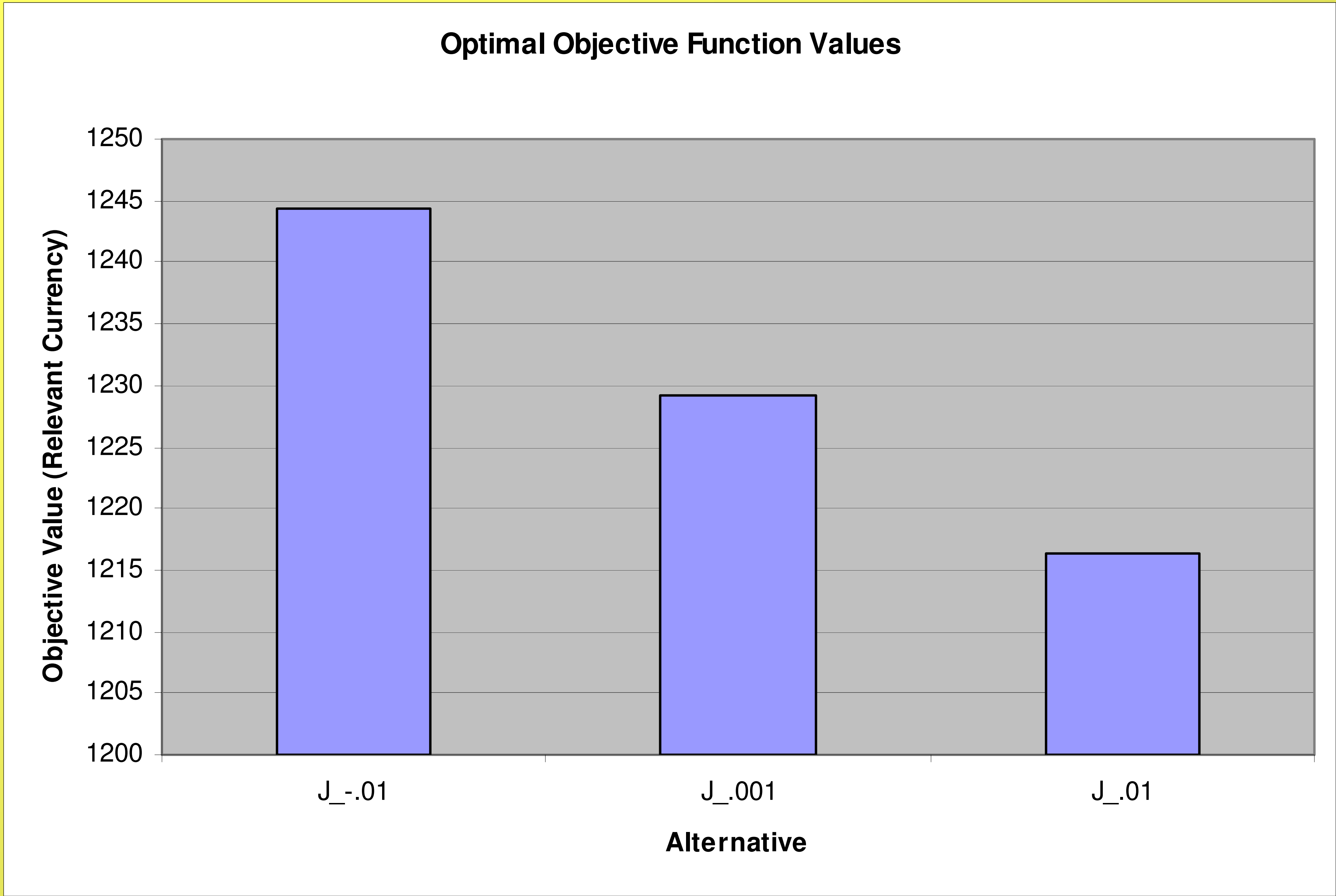
Optimal strategy and dynamic effects of

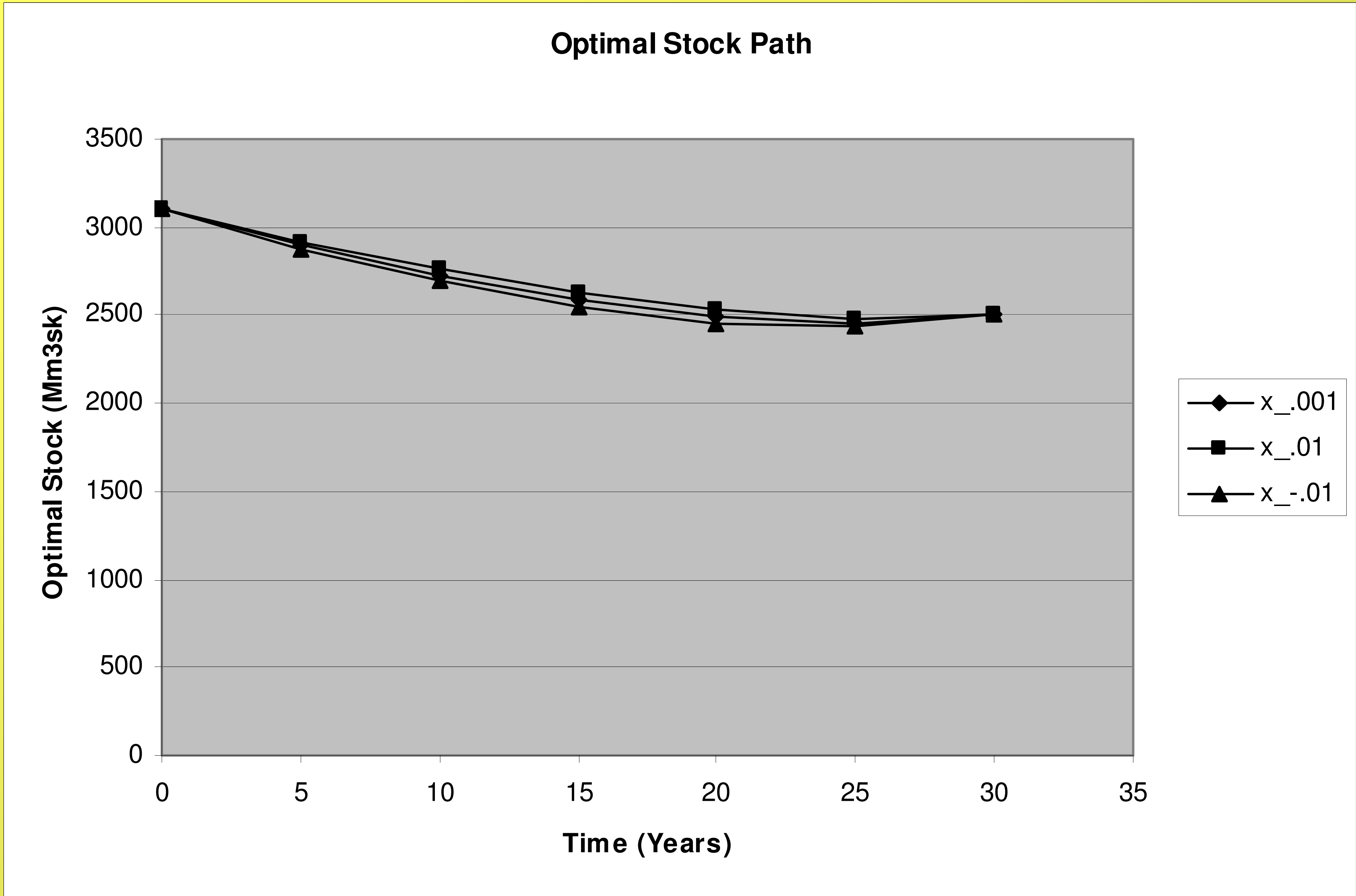
the growth function

t1	t2	r
0	30	.05
f1	f2	
0	0	
k1	k2	k3
1000	0	-3
g0	g1	g2
75	.01	0
x1	x2	
3100	2500	

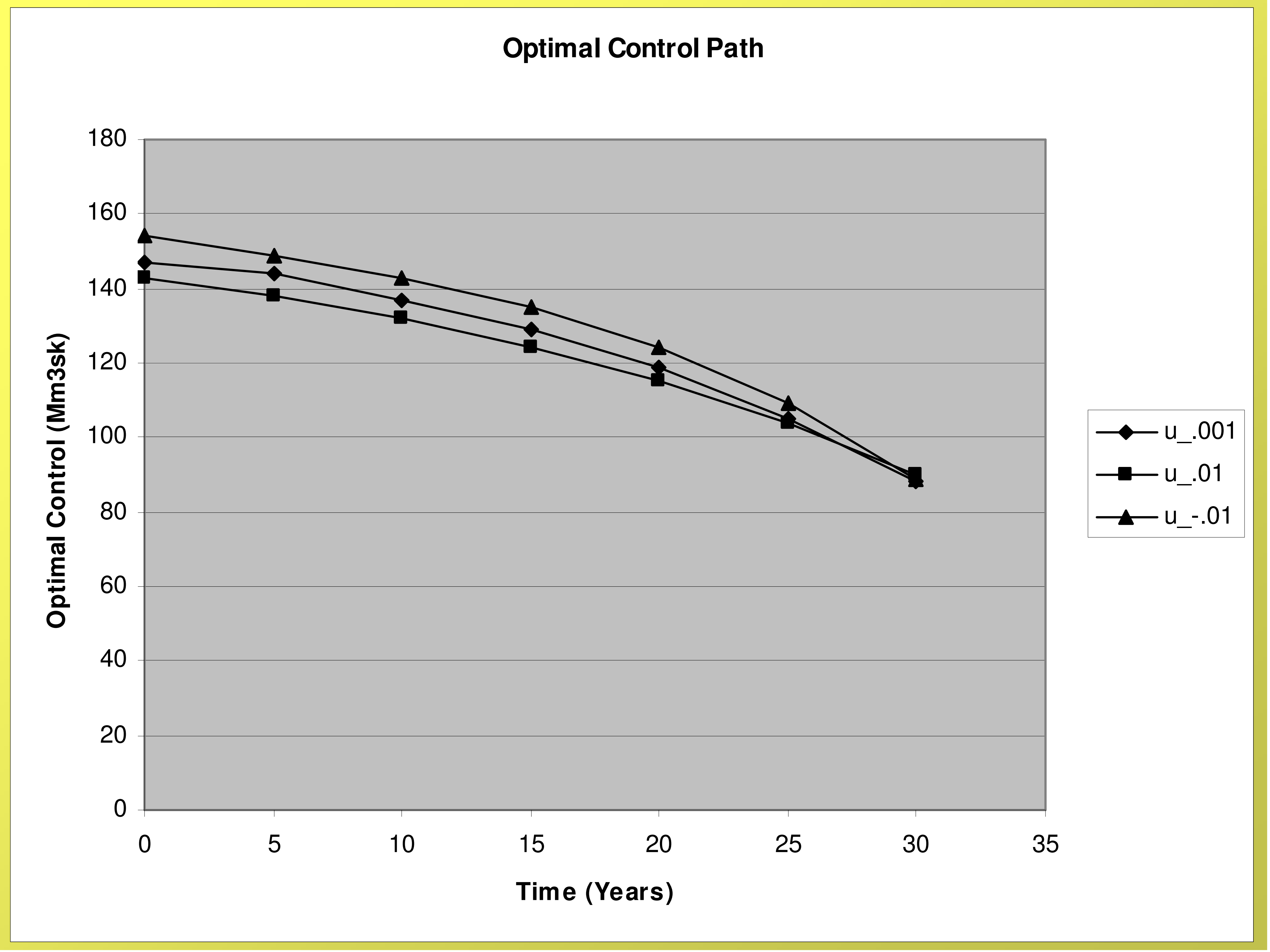
0	30	.05
0	0	
1000	0	-3
102.9	.001	0
3100	2500	

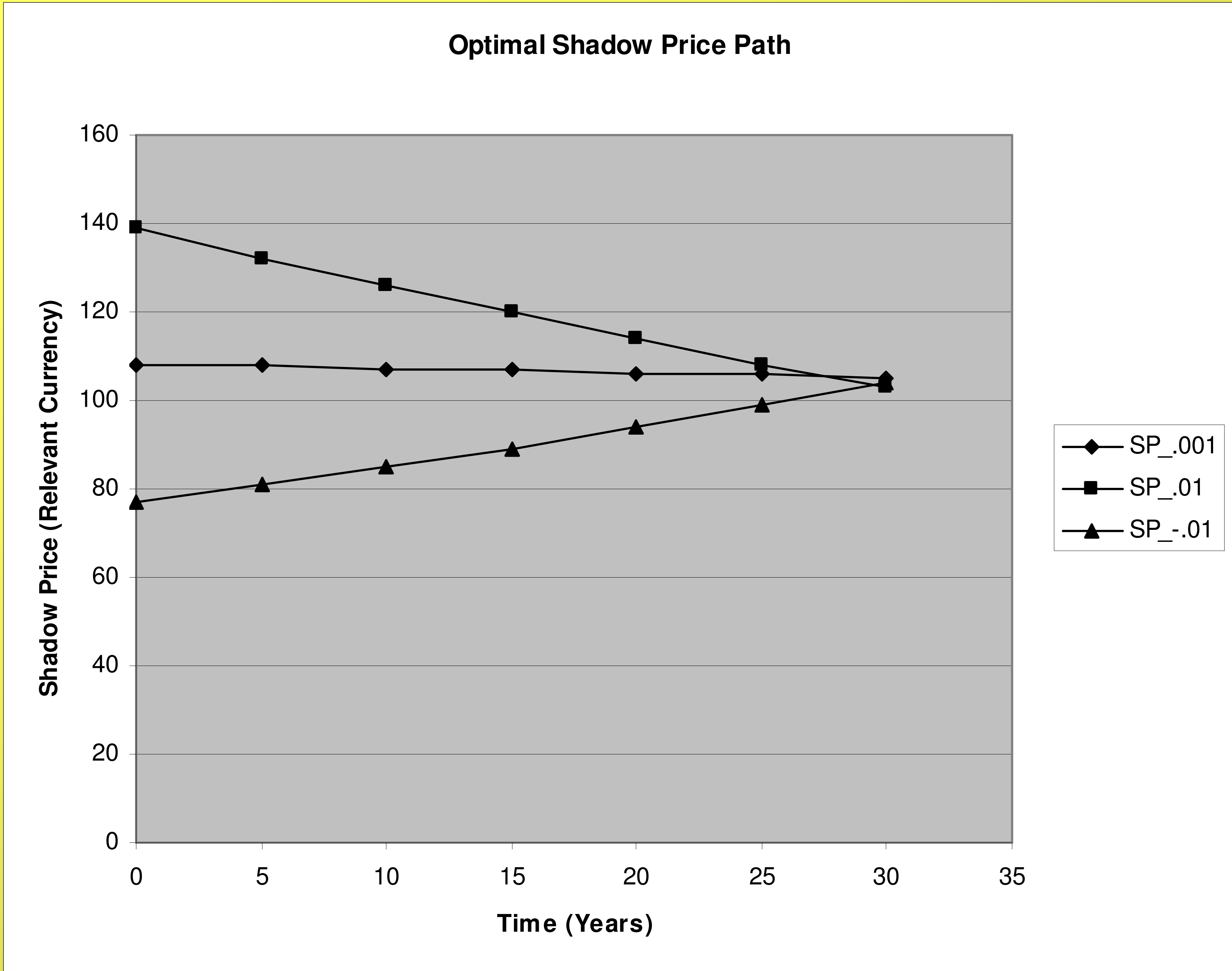
0	30	.05
0	0	
1000	0	-3
137	-.01	0
3100	2500	


Optimal Objective Function Values
Objective Value (Relevant Currency)
1250
1245
1240
1235
1230
1225
1220
1215
1210
1205
1200
J_-.01
J_.001
J_.01
Alternative



Optimal Stock Path
Optimal Stock (Mm3sk)
3500
3000
2500
2000
1500
1000
500
0
0
5
10
15
20
25
30
35
Time (Years)
x_.001
x_.01
x_-.01


Optimal Control Path
Optimal Control (Mm3sk)
Time (Years)
0
20
40
60
80
100
120
140
160
180
0
5
10
15
20
25
30
35
u_.001
u_.01
u_-.01


Optimal Shadow Price Path
Shadow Price (Relevant Currency)
160
140
120
100
80
60
40
20
0
0
5
10
15
20
25
30
35
Time (Years)
SP_.001
SP_.01
SP_-.01