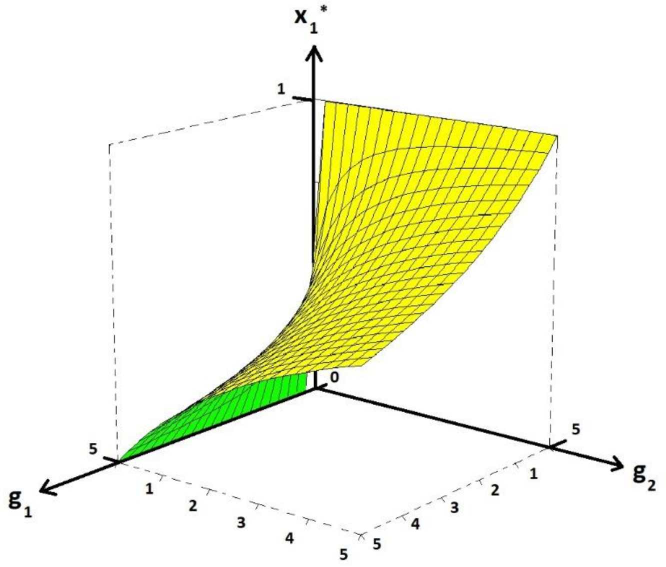
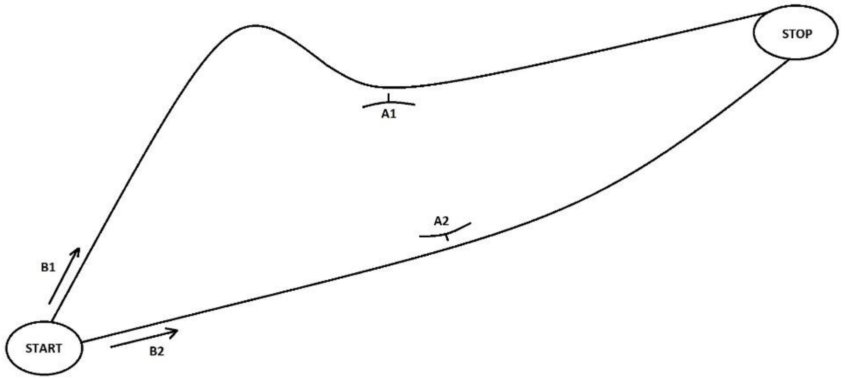
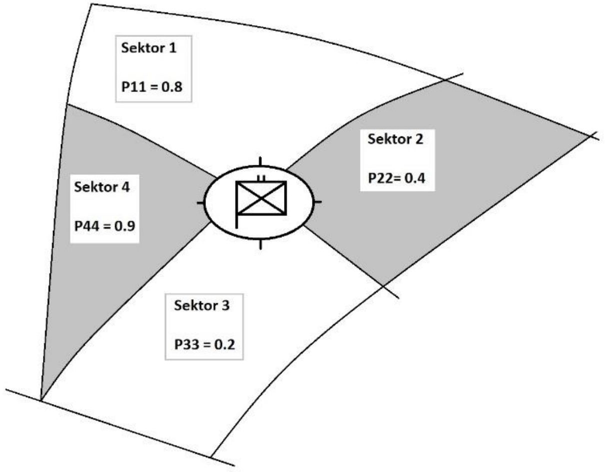
RECENT ADVANCES IN GENERAL GAME THEORY AND APPLICATIONS







Peter Lohmander

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Lecture at Mid Sweden University

Department of Economics, Geography, Law and Tourism (EJT), Sundsvall

2020-01-16 13.00 – 14.30

- In part 1 of this presentation, the two player zero sum games with diagonal game matrixes, TPZSGD, are analyzed.

- Many important applications of this particular class of games are found in military decision problems, in customs and immigration strategies and police work.

- Explicit functions are derived that give the optimal frequences of different decisions and the expected results of relevance to the different decision makers.

- Arbitrary numbers of decision alternatives are covered.

- It is proved that the derived optimal decision frequency formulas correspond to the unique optimization results of the two players.

- It is proved that the optimal solutions, for both players, always lead to a unique completely mixed strategy Nash equilibrium.

- For each player, the optimal frequency of a particular decision is strictly greater than 0 and strictly less than 1.

- With comparative statics analyses, the directions of the changes of optimal decision frequences and expected game values as functions of changes in different parameter values, are determined.

- The signs of the optimal changes of the decision frequences, of the different players, are also determined as functions of risk in different parameter values.

- Furthermore, the directions of changes of the expected optimal value of the game, are determined as functions of risk in the different parameter values.

- Finally, some of the derived formulas are used to confirm earlier game theory results presented in the literature. It is demonstrated that the new functions can be applied to solve common military problems.

- In part 2 of this presentation, four military decision problems, common and relevant to typical army and ranger units, at platoon, company and battalion levels, are described and analysed.

- It is found that fundamental game theory and methods can be used to determine optimal decisions.

- The optimal decisions are derived as mixed strategy Nash equilibria, via manual methods.

- It is found that considerable improvements of the expected outcomes of typical decisions can be obtained in a way that does not require high investment costs.

- It is argued that the methodology to some degree should be included in the education of all Swedish military officers, in particular in the army and ranger units intended for special operations.

- In part 3, stochastic dynamic extensions of part 1 will be defined.

References to this presentation:

- Lohmander, P., **Optimal decisions and expected values in two player zero sum games with diagonal game matrixes, - Explicit functions, general proofs and effects of parameter estimation errors,** ***International Robotics & Automation Journal***, Volume 5, Issue 5, 2019, pages 186-198.

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- Lohmander, P., **Applications and Mathematical Modeling in Operations Research**, In: Cao BY. (ed) Fuzzy Information and Engineering and Decision. IWDS 2016. ***Advances in Intelligent Systems and Computing***, vol 646. Springer, Cham, 2018 Print ISBN 978-3-319-66513-9, Online ISBN 978-3-319-66514-6, eBook Package: Engineering

References to this presentation (continued):

- Lohmander, P., **Applications and mathematical modeling in operations research**, **KEYNOTE**, International Conference on Mathematics and Decision Science, International Center of Optimization and Decision Making & Guangzhou University, Guangzhou, China, September 12-15, 2016 http://www.Lohmander.com/PL_KEYNOTE_MATH_2016.jpg
  http://www.Lohmander.com/PL_ICODM_2016_KEY.pptx
  http://www.Lohmander.com/PL_ICODM_2016_KEY.pdf
  http://www.Lohmander.com/PL_ICODM_2016_KEY_PAPER.pdf
  http://www.Lohmander.com/PL_ICODM_2016_KEY_PAPER.docx
  http://icodm2020.com/en/

Presentation of Peter Lohmander

- Bauer, M., Peter Lohmander, IIASA,
  International Institute for Applied Systems Analysis,
  http://www.iiasa.ac.at/web/home/about/alumni/News/20181204_lohmander.html
  http://www.Lohmander.com/PL_IIASA_18.pdf

References on related topics

- http://www.lohmander.com/Information/Ref.htm

First, we start with some very concrete decision problems, with only 2 and 4 dimensions.

Later, we will generalize the findings to arbitrary numbers of dimensions.

KUNGL
KRIGSVETENSKAPS-
AKADEMIENS
Handlingar och Tidskrift
NR 2/2019

Publicerad sedan 1797

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OF WAR SCIENCES
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Optimal decisions and expected values in two player zero sum games with diagonal game matrixes-explicit functions, general proofs and effects of parameter estimation errors

Abstract

In this paper, the two player zero sum games with diagonal game matrixes, TPZSGD, are analyzed. Many important applications of this particular class of games are found in military decision problems, in customs and immigration strategies and police work. Explicit functions are derived that give the optimal frequences of different decisions and the expected results of relevance to the different decision makers. Arbitrary numbers of decision alternatives are covered. It is proved that the derived optimal decision frequency formulas correspond to the unique optimization results of the two players. It is proved that the optimal solutions, for both players, always lead to a unique completely mixed strategy Nash equilibrium. For each player, the optimal frequency of a particular decision is strictly greater than 0 and strictly less than 1. With comparative statics analyses, the directions of the changes of optimal decision frequences and expected game values as functions of changes in different parameter values, are determined. The signs of the optimal changes of the decision frequences, of the different players, are also determined as functions of risk in different parameter values. Furthermore, the directions of changes of the expected optimal value of the game, are determined as functions of risk in the different parameter values. Finally, some of the derived formulas are used to confirm earlier game theory results presented in the literature. It is demonstrated that the new functions can be applied to solve common military problems.

Keywords: optimal decisions, completely mixed strategy Nash equilibrium, zero sum game theory, stochastic games

Volume 5 Issue 5 - 2019

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Beslutsproblem

Vägval vid uppmarsch och underhållstransporter

Val av plats för eldöverfall vid fördröjningsstrid

Positionering av bevaknings- och stridspatruller vid stabsplats

Val av utgångsgruppering för spaning mot, och störande av, fientlig stabsplats

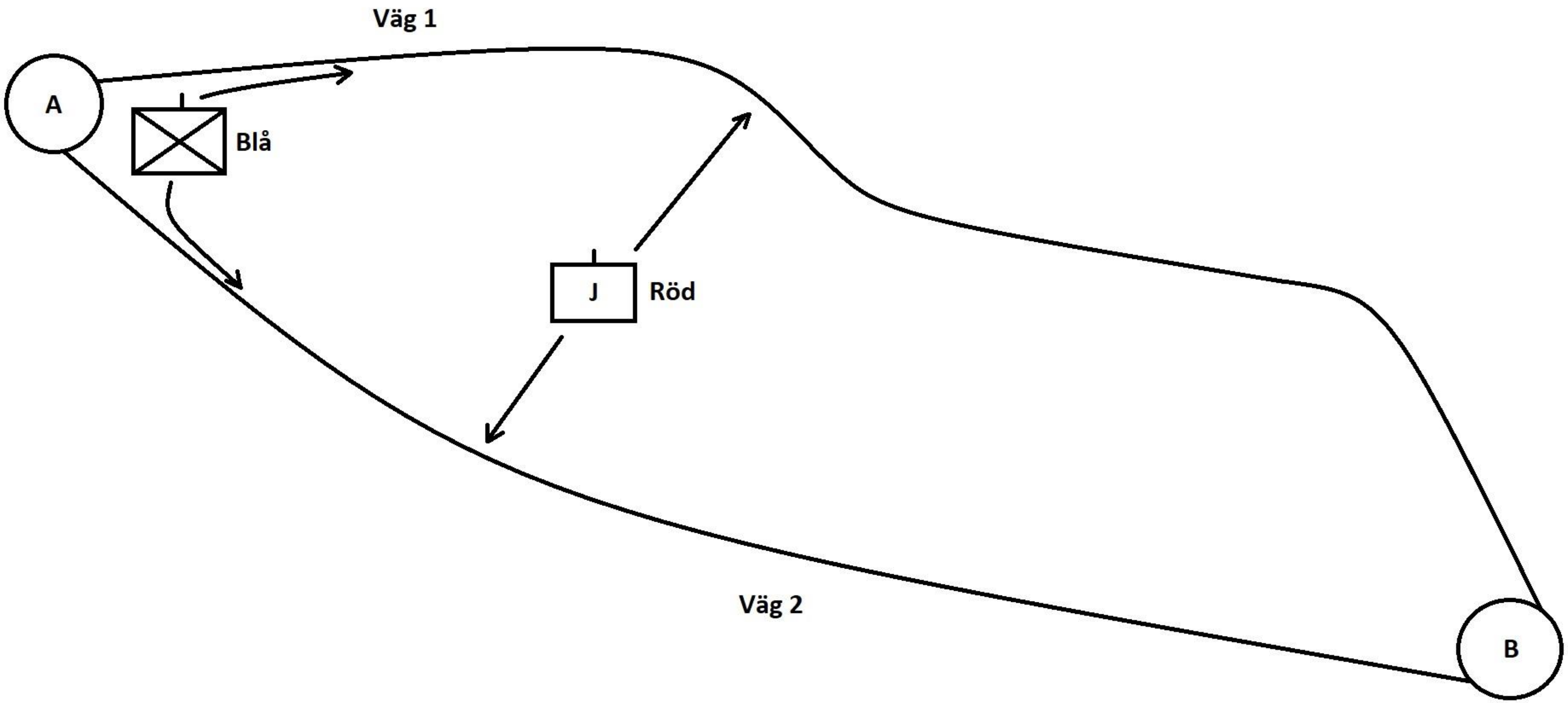
Decision Problems

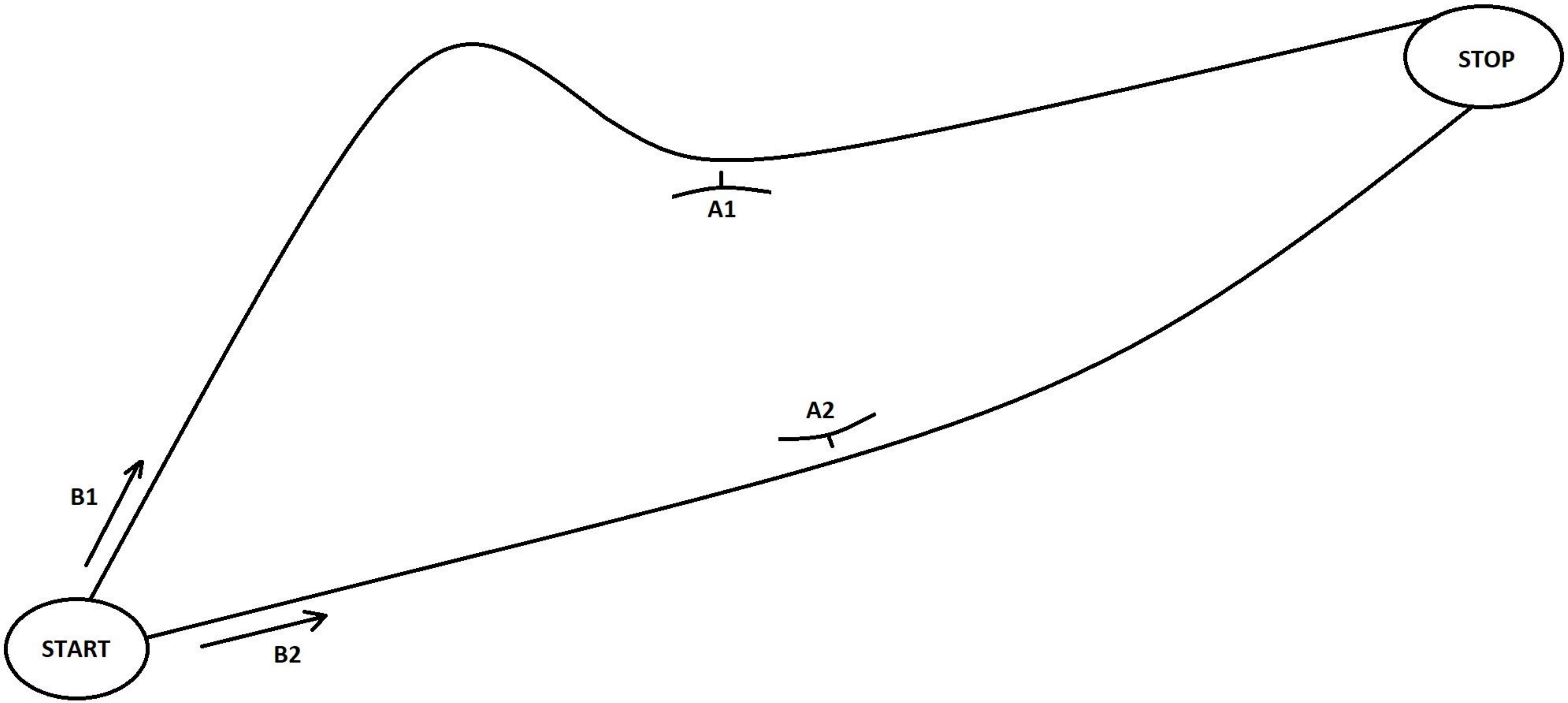
The selection of roads for transport when enemy forces may prepare attacks along different roads with different expected outcomes,

The selection of roads where attacks on enemy transports should be prepared,

The positioning of guard squads and

The positioning of intelligence, reconaissance and sabotage groups.


Väg 1
A
Blå
J
Röd
Väg 2
B


STOP
A1
A2
B1
START
B2

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix}$$

$$\min E$$

$$s.t.$$

$$E \geq c_{11} y_1 + 0 y_2 \quad (if\ A_1)$$

$$E \geq 0 y_1 + c_{22} y_2 \quad (if\ A_2)$$

$$1 = y_1 + y_2$$

$$0 \leq y_1$$

$$0 \leq y_2$$

$$\min E$$

$$s.t.$$

$$E \geq c_{11} y_1 \qquad (if\ A_1)$$

$$E \geq c_{22}(1 - y_1) \qquad (if\ A_2)$$

$$\min E$$

$$s.t.$$

$$E \geq c_{11} y_1 \qquad \left(if\ A_1 \right)$$

$$E \geq c_{22} - c_{22} y_1 \qquad \left(if\ A_2 \right)$$

Special case:

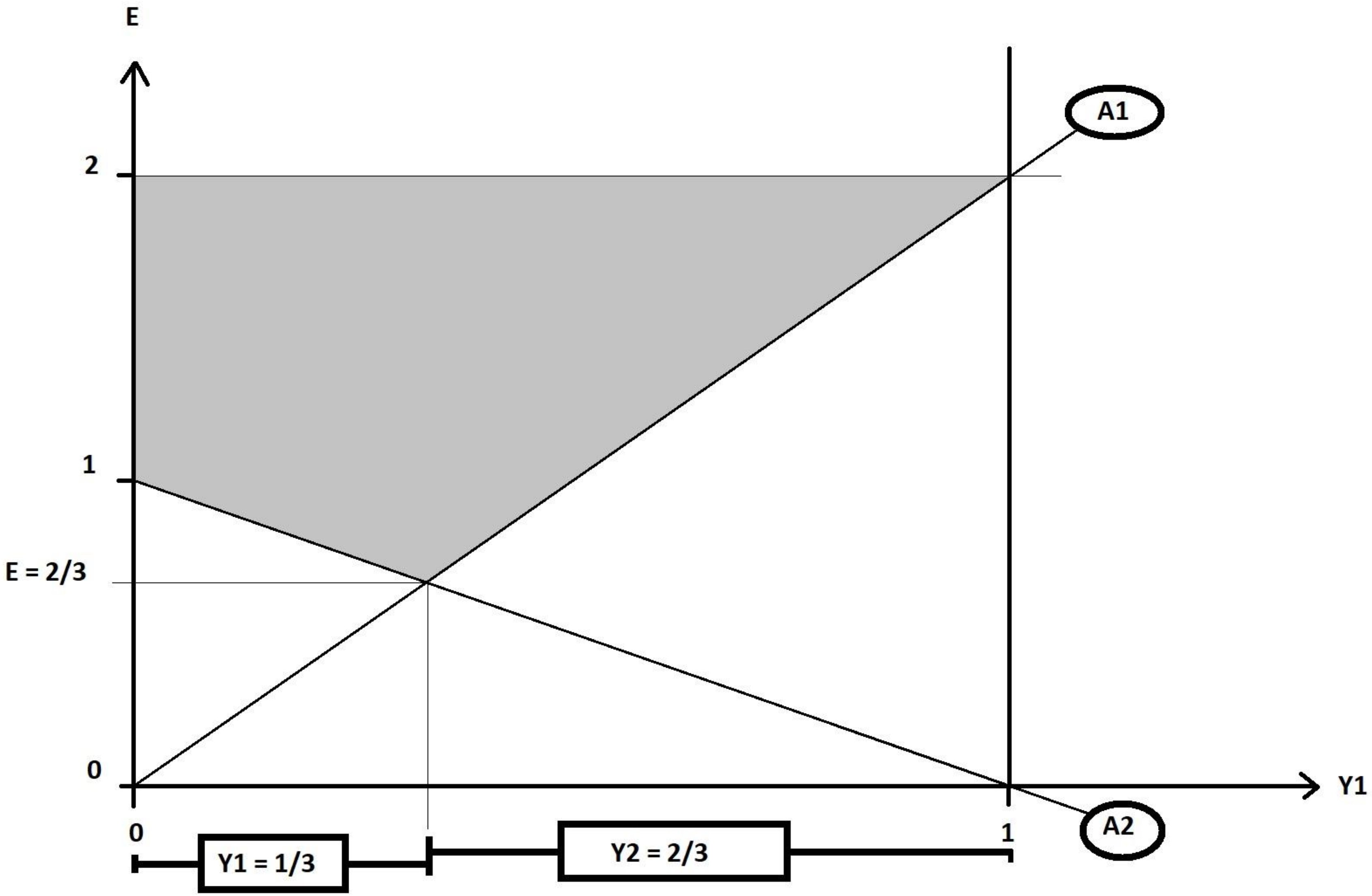
$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\min E$$

$$s.t.$$

$$E \geq 2y_1 \qquad (if\ A_1)$$

$$E \geq 1 - y_1 \qquad (if\ A_2)$$


E
2
1
E = 2/3
0
A1
A2
Y1
0
1
Y1 = 1/3
Y2 = 2/3

$$\max E$$

$$s.t.$$

$$E \le c_{11}x_1 + 0x_2 \quad \left(if\ \ B_1\right)$$

$$E \le 0x_1 + c_{22}x_2 \quad \left(if\ \ B_2\right)$$

$$1 = x_1 + x_2$$

$$0 \le x_1$$

$$0 \le x_2$$

$$\max E$$

$$s.t.$$

$$E \le c_{11}x_1 \quad \left(if\ B_1\right)$$

$$E \le c_{22}(1 - x_1) \quad \left(if\ B_2\right)$$

$$\max E$$

$$s.t.$$

$$E \le c_{11} x_1 \qquad \left(if \ B_1 \right)$$

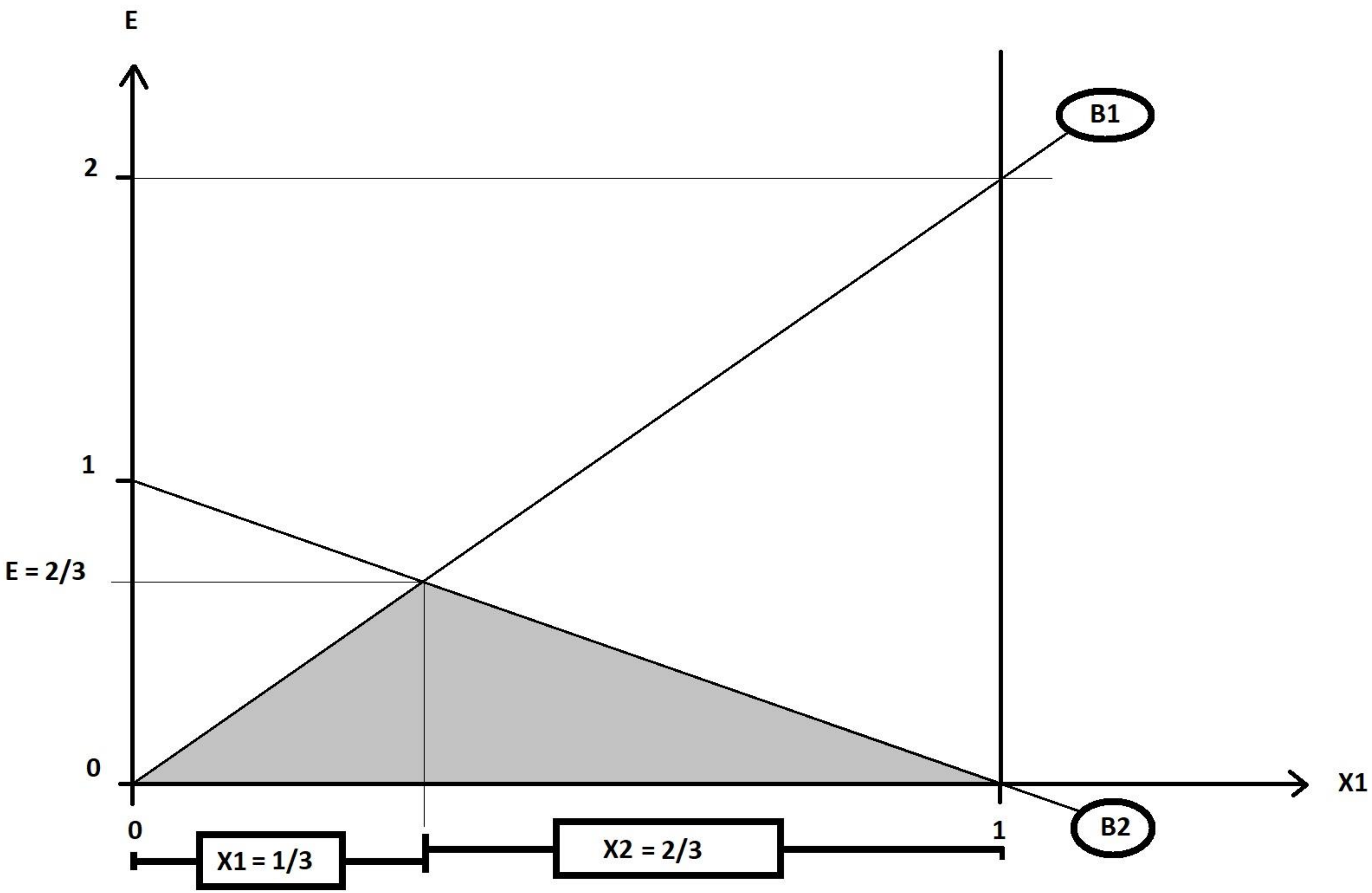
$$E \le c_{22} - c_{22} x_1 \qquad \left(if \ B_2 \right)$$

$$\max E$$

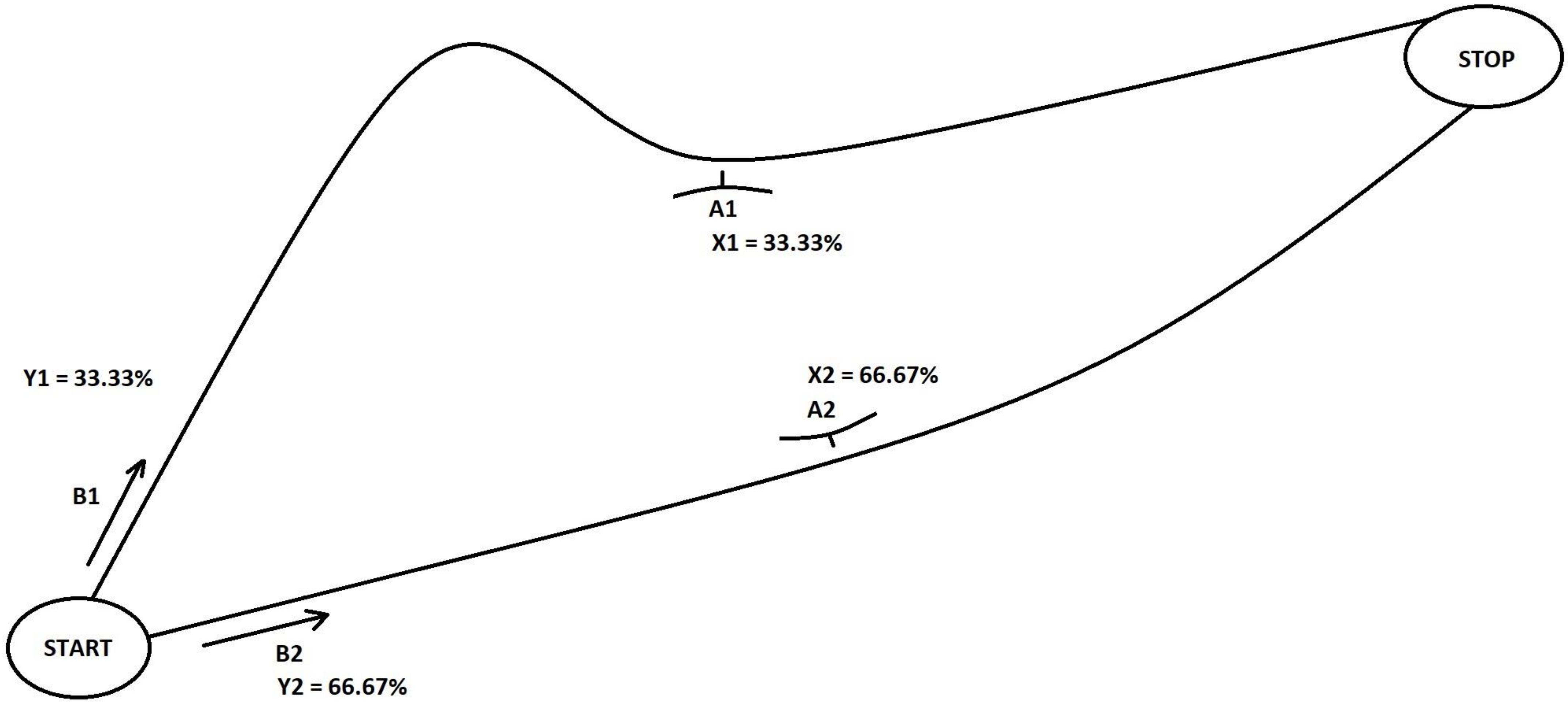
$$s.t.$$

$$E \leq 2x_1 \qquad (if\ B_1)$$

$$E \leq 1 - 1x_1 \qquad (if\ B_2)$$


E
2
1
E = 2/3
0
B1
B2
X1
0
1
X1 = 1/3
X2 = 2/3


New Results


STOP
A1
X1 = 33.33%
Y1 = 33.33%
X2 = 66.67%
A2
B1
START
B2
Y2 = 66.67%

Observation:

If we can be sure that, in optimum, all decisions have stricly positive probabilities, then we know that:

$$E = x_1 c_{11} = x_2 c_{22}$$

Then, if the number of possible decisions is 2, we have:

$$E = x_1 c_{11} = (1 - x_1) c_{22}$$

$$x_1 c_{11} = c_{22} - c_{22} x_1$$

$$x_1 (c_{11} + c_{22}) = c_{22}$$

$$x_1 = \frac{c_{22}}{c_{11} + c_{22}}$$

$$x_2 = (1 - x_1) = \left(1 - \frac{c_{22}}{c_{11} + c_{22}}\right)$$

$$x_2 = \left(\frac{c_{11} + c_{22}}{c_{11} + c_{22}} - \frac{c_{22}}{c_{11} + c_{22}}\right)$$

$$x_2 = \frac{c_{11}}{c_{11} + c_{22}}$$

Particular decision rules

Observation:

When there are exactly two possible decisions, and the optimal probabilities are strictly positive, we may calculate the expected value of the game in two ways. The results are identical.

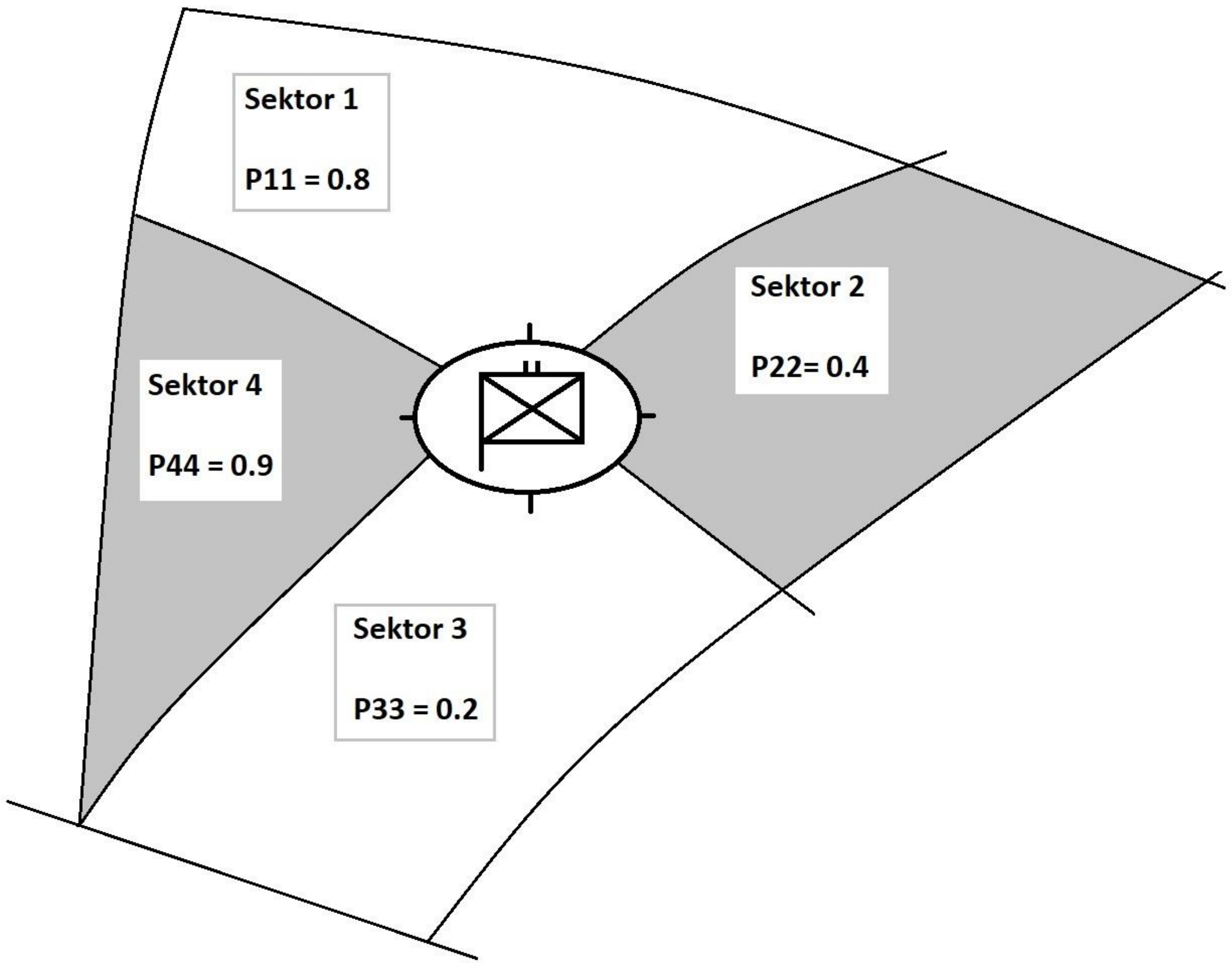
$$E = x_1 c_{11} = x_2 c_{22}$$

$$x_1 = \frac{c_{22}}{c_{11} + c_{22}}$$

$$x_2 = \frac{c_{11}}{c_{11} + c_{22}}$$

$$E = x_1 c_{11} = \frac{c_{11} c_{22}}{c_{11} + c_{22}}$$

$$E = x_2 c_{22} = \frac{c_{11} c_{22}}{c_{11} + c_{22}}$$


Sektor 1
P11 = 0.8
Sektor 2
P22= 0.4
Sektor 4
P44 = 0.9
Sektor 3
P33 = 0.2

$$\max \quad E$$

$$s.t.$$

$$E \le p_{11} x_1 \qquad (if\ S_1)$$

$$E \le p_{22} x_2 \qquad (if\ S_2)$$

$$E \le p_{33} x_3 \qquad (if\ S_3)$$

$$E \le p_{44} x_4 \qquad (if\ S_4)$$

$$1 = x_1 + x_2 + x_3 + x_4$$

$$x_1 \ge 0;\ x_2 \ge 0;\ x_3 \ge 0;\ x_4 \ge 0$$

If we know that the optimal frequences of all decision are strictly positive, then:

$$E = p_{11}x_1$$

$$E = p_{22}x_2$$

$$E = p_{33}x_3$$

$$E = p_{44}x_4$$

$$E = p_{11}x_1 = p_{22}x_2 = p_{33}x_3 = p_{44}x_4$$

$$x_1 = \frac{E}{p_{11}}$$

$$x_2 = \frac{E}{p_{22}}$$

$$x_3 = \frac{E}{p_{33}}$$

$$x_4 = \frac{E}{p_{44}}$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

$$\frac{E}{p_{11}} + \frac{E}{p_{22}} + \frac{E}{p_{33}} + \frac{E}{p_{44}} = 1$$

$$\frac{1}{p_{11}}+\frac{1}{p_{22}}+\frac{1}{p_{33}}+\frac{1}{p_{44}}=\frac{1}{E}$$

$$\frac{p_{22}p_{33}p_{44}+p_{11}p_{33}p_{44}+p_{11}p_{22}p_{44}+p_{11}p_{22}p_{33}}{p_{11}p_{22}p_{33}p_{44}}=\frac{1}{E}$$

$$\frac{p_{22}p_{33}p_{44} + p_{11}p_{33}p_{44} + p_{11}p_{22}p_{44} + p_{11}p_{22}p_{33}}{p_{11}p_{22}p_{33}p_{44}} = \frac{1}{E}$$

$$\frac{0.4 \bullet 0.2 \bullet 0.9 + 0.8 \bullet 0.2 \bullet 0.9 + 0.8 \bullet 0.4 \bullet 0.9 + 0.8 \bullet 0.4 \bullet 0.2}{0.8 \bullet 0.4 \bullet 0.2 \bullet 0.9} = \frac{1}{E}$$

$$E = \frac{0.8 \bullet 0.4 \bullet 0.2 \bullet 0.9}{0.4 \bullet 0.2 \bullet 0.9 + 0.8 \bullet 0.2 \bullet 0.9 + 0.8 \bullet 0.4 \bullet 0.9 + 0.8 \bullet 0.4 \bullet 0.2}$$

$$E \approx 0.10140845$$

$$E \approx 10\%$$

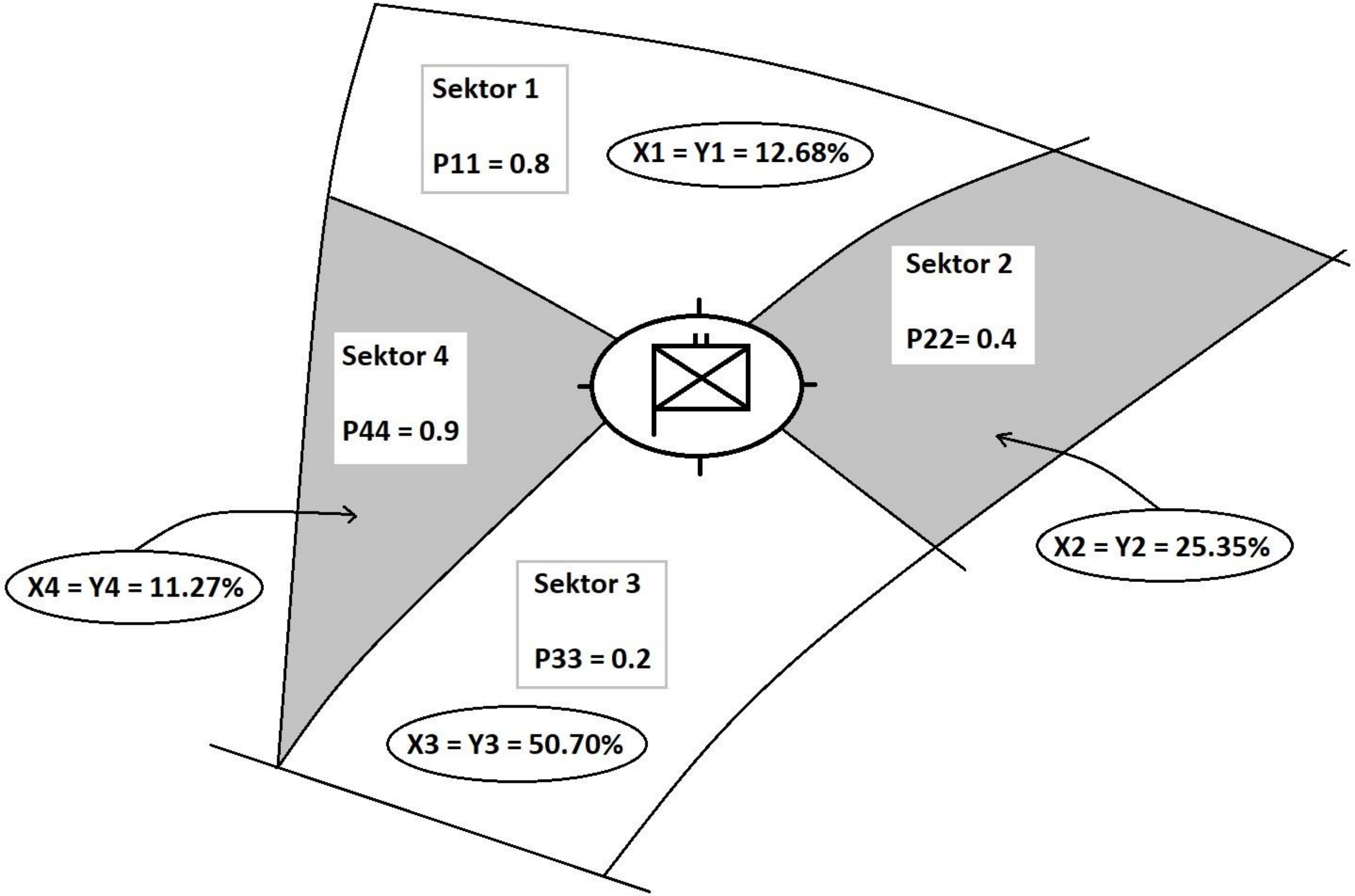
$$x_1 = \frac{E}{p_{11}} = \frac{E}{0.8} \approx 12.68\%$$

$$x_2 = \frac{E}{p_{22}} = \frac{E}{0.4} \approx 25.35\%$$

$$x_3 = \frac{E}{p_{33}} = \frac{E}{0.2} \approx 50.70\%$$

$$x_4 = \frac{E}{p_{44}} = \frac{E}{0.9} \approx 11.27\%$$


New Results


Sektor 1
P11 = 0.8
X1 = Y1 = 12.68%
Sektor 2
P22= 0.4
X2 = Y2 = 25.35%
Sektor 4
P44 = 0.9
X4 = Y4 = 11.27%
Sektor 3
P33 = 0.2
X3 = Y3 = 50.70%

$$\min \quad E$$

$$s.t.$$

$$E \geq p_{11} y_1 \qquad (if \ S_1)$$

$$E \geq p_{22} y_2 \qquad (if \ S_2)$$

$$E \geq p_{33} y_3 \qquad (if \ S_3)$$

$$E \geq p_{44} y_4 \qquad (if \ S_4)$$

$$1 = y_1 + y_2 + y_3 + y_4$$

$$y_1 \geq 0; \ y_2 \geq 0; \ y_3 \geq 0; \ y_4 \geq 0$$

If we know that the optimal frequences of all decision are strictly positive, then:

$$E = p_{11} y_1$$

$$E = p_{22} y_2$$

$$E = p_{33} y_3$$

$$E = p_{44} y_4$$

$$E = p_{11}y_1 = p_{22}y_2 = p_{33}y_3 = p_{44}y_4$$

$$y_1 = \frac{E}{p_{11}} = x_1$$

$$y_2 = \frac{E}{p_{22}} = x_2$$

$$y_3 = \frac{E}{p_{33}} = x_3$$

$$y_4 = \frac{E}{p_{44}} = x_4$$

$$y_1 + y_2 + y_3 + y_4 = 1$$

$$\frac{E}{p_{11}} + \frac{E}{p_{22}} + \frac{E}{p_{33}} + \frac{E}{p_{44}} = 1$$

$$\frac{E}{p_{11}} + \frac{E}{p_{22}} + \frac{E}{p_{33}} + \frac{E}{p_{44}} = 1$$

$$\frac{1}{p_{11}} + \frac{1}{p_{22}} + \frac{1}{p_{33}} + \frac{1}{p_{44}} = \frac{1}{E}$$

$$\frac{1}{p_{11}}+\frac{1}{p_{22}}+\frac{1}{p_{33}}+\frac{1}{p_{44}}=\frac{1}{E}$$

$$\frac{p_{22}p_{33}p_{44}+p_{11}p_{33}p_{44}+p_{11}p_{22}p_{44}+p_{11}p_{22}p_{33}}{p_{11}p_{22}p_{33}p_{44}}=\frac{1}{E}$$

$$\frac{p_{22}p_{33}p_{44} + p_{11}p_{33}p_{44} + p_{11}p_{22}p_{44} + p_{11}p_{22}p_{33}}{p_{11}p_{22}p_{33}p_{44}} = \frac{1}{E}$$

$$\frac{0.4 \bullet 0.2 \bullet 0.9 + 0.8 \bullet 0.2 \bullet 0.9 + 0.8 \bullet 0.4 \bullet 0.9 + 0.8 \bullet 0.4 \bullet 0.2}{0.8 \bullet 0.4 \bullet 0.2 \bullet 0.9} = \frac{1}{E}$$

$$E = \frac{0.8 \bullet 0.4 \bullet 0.2 \bullet 0.9}{0.4 \bullet 0.2 \bullet 0.9 + 0.8 \bullet 0.2 \bullet 0.9 + 0.8 \bullet 0.4 \bullet 0.9 + 0.8 \bullet 0.4 \bullet 0.2}$$

$$E \approx 0.10140845$$

$$E \approx 10\%$$

$$y_1 = \frac{E}{p_{11}} = \frac{E}{0.8} \approx 12.68\%$$

$$y_2 = \frac{E}{p_{22}} = \frac{E}{0.4} \approx 25.35\%$$

$$y_3 = \frac{E}{p_{33}} = \frac{E}{0.2} \approx 50.70\%$$

$$y_4 = \frac{E}{p_{44}} = \frac{E}{0.9} \approx 11.27\%$$

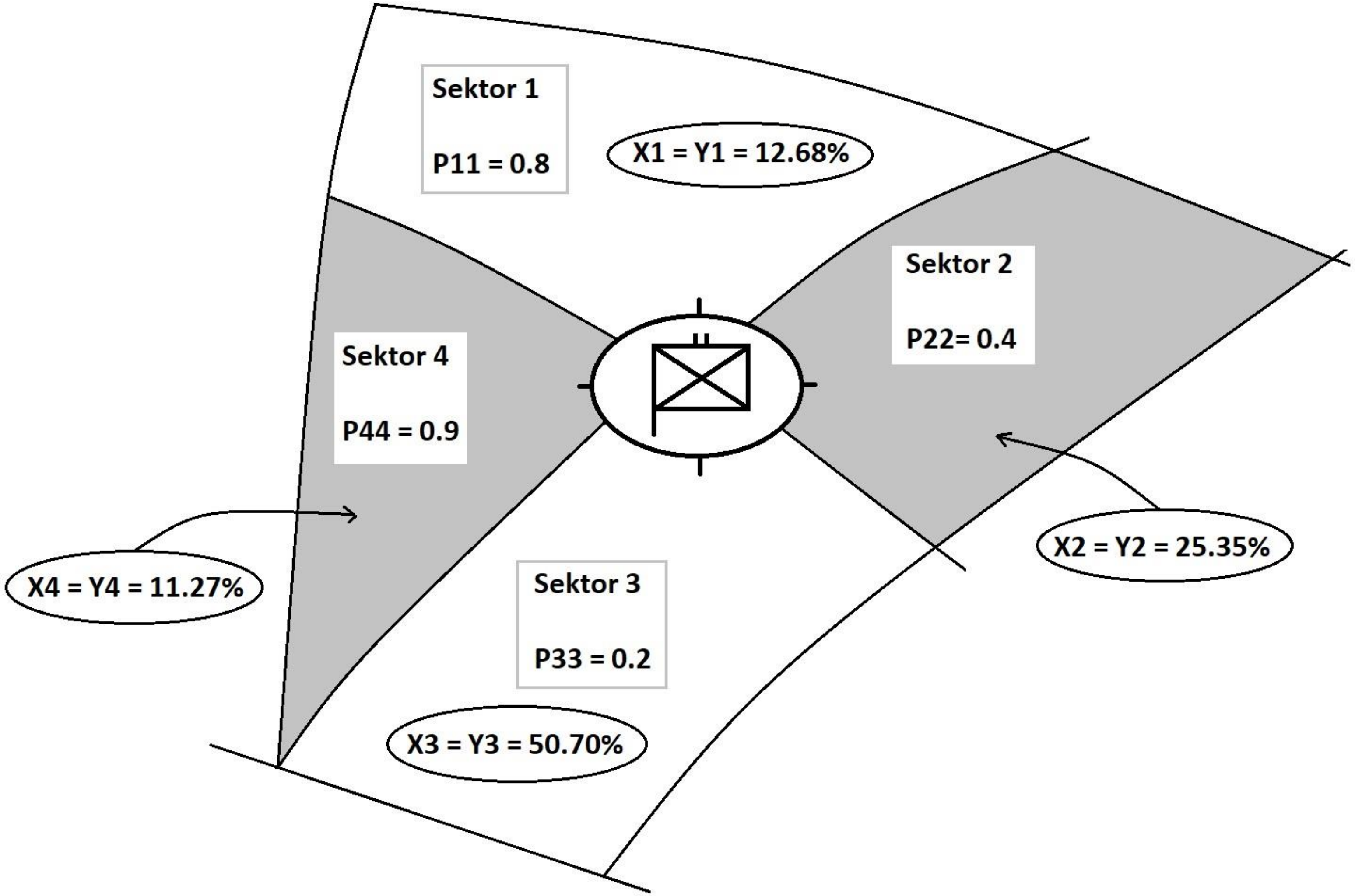
$$y_1 = x_1$$

$$y_2 = x_2$$

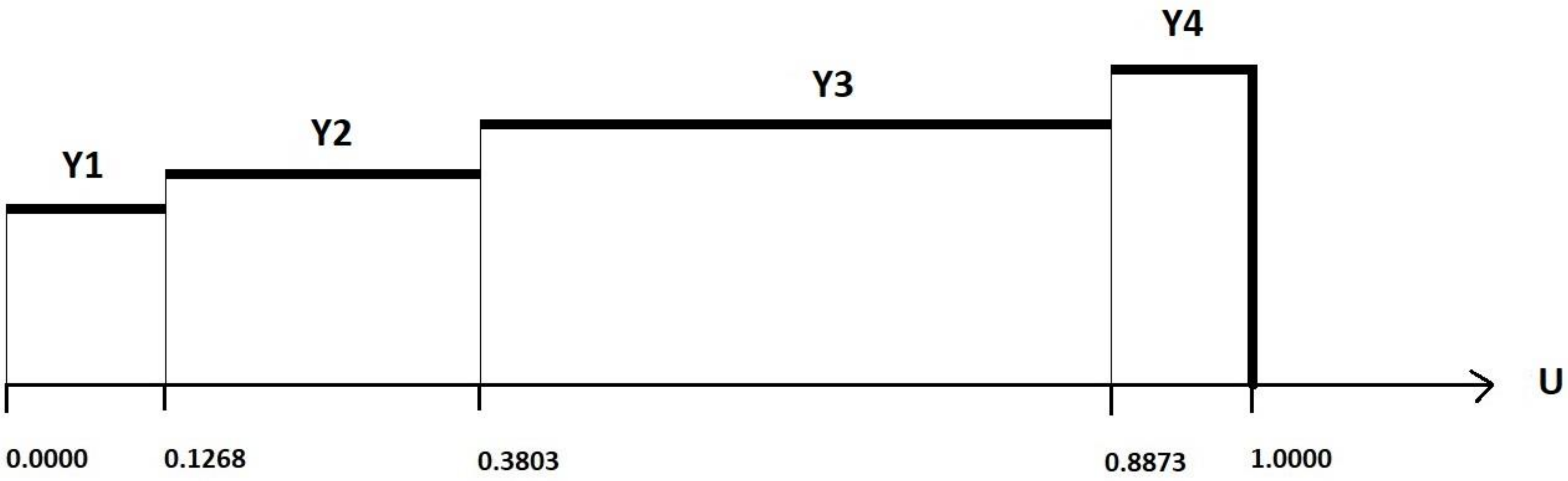
$$y_3 = x_3$$

$$y_4 = x_4$$


New Results


Sektor 1
P11 = 0.8
X1 = Y1 = 12.68%
Sektor 2
P22= 0.4
X2 = Y2 = 25.35%
Sektor 3
P33 = 0.2
X3 = Y3 = 50.70%
Sektor 4
P44 = 0.9
X4 = Y4 = 11.27%


X1
X2
X3
X4
S
0.0000
0.1268
0.3803
0.8873
1.0000


Y4
Y3
Y2
Y1
U
0.0000
0.1268
0.3803
0.8873
1.0000


Test

Is this correct?

$$E = x_1 y_1 p_{11} + x_2 y_2 p_{22} + x_3 y_3 p_{33} + x_4 y_4 p_{44}$$

$$E \approx 0.10140845$$

YES!


New General Theory

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Research Article

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Optimal decisions and expected values in two player zero sum games with diagonal game matrixes-explicit functions, general proofs and effects of parameter estimation errors

Abstract

In this paper, the two player zero sum games with diagonal game matrixes, TPZSGD, are analyzed. Many important applications of this particular class of games are found in military decision problems, in customs and immigration strategies and police work. Explicit functions are derived that give the optimal frequences of different decisions and the expected results of relevance to the different decision makers. Arbitrary numbers of decision alternatives are covered. It is proved that the derived optimal decision frequency formulas correspond to the unique optimization results of the two players. It is proved that the optimal solutions, for both players, always lead to a unique completely mixed strategy Nash equilibrium. For each player, the optimal frequency of a particular decision is strictly greater than 0 and strictly less than 1. With comparative statics analyses, the directions of the changes of optimal decision frequences and expected game values as functions of changes in different parameter values, are determined. The signs of the optimal changes of the decision frequences, of the different players, are also determined as functions of risk in different parameter values. Furthermore, the directions of changes of the expected optimal value of the game, are determined as functions of risk in the different parameter values. Finally, some of the derived formulas are used to confirm earlier game theory results presented in the literature. It is demonstrated that the new functions can be applied to solve common military problems.

Keywords: optimal decisions, completely mixed strategy Nash equilibrium, zero sum game theory, stochastic games

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$$c_{ij}\Big|_{i\neq j}=0, i=1,\dots,n,\ j=1,2,\dots,n \tag{2.1}$$

$$c_{ij}\Big|_{i=j}=g_i>0\ ,\ i=1,\dots,n,\ j=1,2,\dots,n \tag{2.2}$$

$$\max x_0 \tag{2.1.1}$$

s.t.

$$\sum_{i=1}^{n} x_i \leq 1 \tag{2.1.2}$$

$$x_0 \leq g_i x_i\ ,\ i=1,\dots,n \tag{2.1.3}$$

$$x_i \geq 0\ ,\ i=1,\dots,n \tag{2.1.4}$$

Let λ_i denote dual variables. The following Lagrange function is defined:

$$L = x_0 + \lambda_0\left(1 - \sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} \lambda_i\left(g_i x_i - x_0\right) \quad (2.1.5)$$

The following derivatives will be needed in the proceeding analysis:

$$\frac{dL}{d\lambda_0} = 1 - \sum_{i=1}^{n} x_i \geq 0 \quad (2.1.6)$$

$$\frac{dL}{d\lambda_i} = g_i x_i - x_0 \geq 0\ ,\ i = 1,\dots,n \quad (2.1.7)$$

$$\frac{dL}{dx_0} = 1 - \sum_{i=1}^{n} \lambda_i \leq 0 \quad (2.1.8)$$

$$\frac{dL}{dx_i} = \lambda_i g_i - \lambda_0 \leq 0\ ,\ i = 1,\dots,n \quad (2.1.9)$$

Karush Kuhn Tucker conditions in general problems

In general problems, we may have different numbers of decision variables and constraints. Furthermore, the elements $c_{ij}\big|_{i \neq j}$ are not necessarily zero (Table 1).

Table 1 Karush Kuhn Tucker conditions in general maximization problems

$\lambda_i \geq 0 \; \forall i$	$\dfrac{dL}{d\lambda_i} \geq 0 \; \forall i$	$\lambda_i \dfrac{dL}{d\lambda_i} = 0 \; \forall i$
$x_j \geq 0 \; \forall j$	$\dfrac{dL}{dx_j} \leq 0 \; \forall j$	$x_j \dfrac{dL}{dx_j} = 0 \; \forall j$

Particular conditions in problems that satisfy (2.1) and (2.2)

Note that in these problems, $i = j$ in all relevant constraints.

$$\lambda_i \geq 0 \ \forall i \tag{2.1.10}$$

$$\frac{dL}{d\lambda_i} \geq 0 \ \forall i \tag{2.1.11}$$

$$\lambda_i \frac{dL}{d\lambda_i} = 0 \ \forall i \tag{2.1.12}$$

$$x_i \geq 0 \ \forall i \tag{2.1.13}$$

$$\frac{dL}{dx_i} \leq 0 \ \forall i \tag{2.1.14}$$

$$x_i \frac{dL}{dx_i} = 0 \ \forall i \tag{2.1.15}$$

Proof 1: Proof that $x_0^* > 0$:

(2.1.2) and (2.1.4) make it feasible to let $x_i > 0$, $i = 1, \ldots n$.

(2.2) says that $g_i > 0$, $i = 1, 2, \ldots, n$.

When $g_i x_i > 0, i = 1, \ldots n$, (2.1.3) makes it feasible to let $x_0 > 0$.

(2.1.1) states that we want to maximize x_0 . Let stars indicate optimal values.

Hence, when optimal decisions are taken, $x_0 = x_0^* > 0$.

Proof 2: Proof that $x_i^* > 0$, $i = 1,\ldots,n$:

(2.1.7) says that $\frac{dL}{d\lambda_i} = g_i x_i - x_0 \geq 0$, $i = 1,\ldots,n$

Proof 1 states that $x_0 > 0$. (2.2) says that $g_i > 0$, $i = 1,\ldots,n$.

$x_i \geq \frac{x_0}{g_i} > 0$, $i = 1,\ldots,n$.

Hence, $x_i = x_i^* > 0$, $i = 0,\ldots,n$.

Proof 3: Proof that $\lambda_i^*, i = 0,\ldots,n$ **can be determined from a linear equation system.**

$$\left(x_i > 0\,,\, i = 0,\ldots,n\right) \wedge \; (2.1.15) \Rightarrow \left\{\frac{dL}{dx_0} = 0\,; \quad \frac{dL}{dx_i} = 0\,,\, i = 1,\ldots,n\right\}$$

$$= \left\{(2.1.16) \wedge (2.1.17)\right\}.$$

$$\frac{dL}{dx_0} = 1 - \sum_{i=1}^{n} \lambda_i = 0 \qquad (2.1.16)$$

$$\frac{dL}{dx_i} = \lambda_i g_i - \lambda_0 = 0\,,\, i = 1,\ldots,n \qquad (2.1.17)$$

Proof 4: Proof that $\lambda_i^* > 0$, $i = 0,\ldots,n$.

(2.1.16) $\Rightarrow \exists i\big|_{i>0,\lambda_i>0}$.

Hence, at least for one strictly positive value i, λ_i is strictly greater than zero.

$\left(\exists i\big|_{i>0,\lambda_i>0}\right) \wedge \left(g_i > 0\,, i = 1,\ldots,n\right) \wedge$ (2.1.17) $\Rightarrow \lambda_0 > 0$.

$$\lambda_0 > 0 \qquad (2.1.18)$$

(2.1.17) $\wedge \left(g_i > 0\,, i = 1,\ldots,n\right) \wedge$ (2.1.18) $\Rightarrow \left(\lambda_i > 0\,, i = 1,\ldots,n\right)$

$$\lambda_i > 0\,,\; i = 1,\ldots,n \qquad (2.1.19)$$

(2.1.18) $\wedge$ (2.1.19) $\Rightarrow \left(\lambda_i > 0\,, i = 0,\ldots,n\right)$

$$\lambda_i^* > 0\,, i = 0,\ldots,n \qquad (2.1.20)$$

Proof 5: Proof that x_i^*, $i = 1,\ldots,n$, can be determined from a linear equation system.

$$\left(\lambda_i > 0\,,\, i = 0,\ldots,n\right) \wedge \ (2.1.12) \ \Rightarrow$$

$$\left\{ \frac{dL}{d\lambda_0} = 0\,; \quad \frac{dL}{d\lambda_i} = 0\,,\, i = 1,\ldots,n \right\} = \left\{ (2.1.21) \wedge (2.1.22) \right\}.$$

$$\frac{dL}{d\lambda_0} = 1 - \sum_{i=1}^{n} x_i = 0 \qquad (2.1.21)$$

$$\frac{dL}{d\lambda_i} = g_i x_i - x_0 = 0\,,\, i = 1,\ldots,n \qquad (2.1.22)$$

Determination of explicit equations that give all values: x_i^* , $i=0,\dots,n$:

(2.1.22) $\Rightarrow$ (2.1.23).

$$x_i = \frac{x_0}{g_i}, i=1,\dots,n \tag{2.1.23}$$

(2.1.21) $\Rightarrow$ (2.1.24).

$$\sum_{i=1}^{n} x_i = 1 \tag{2.1.24}$$

$$\sum_{i=1}^{n} \frac{x_0}{g_i} = 1 \tag{2.1.25}$$

$$\sum_{i=1}^{n} \frac{1}{g_i} = \frac{1}{x_0} \tag{2.1.26}$$

$$x_0 = \frac{1}{\sum_{i=1}^{n} \frac{1}{g_i}} \tag{2.1.27}$$

$$x_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \tag{2.1.28}$$

$$x_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, i=1,\dots,n \tag{2.1.29}$$

Determination of explicit equations that give all values: λ_i^*, $i = 0,\dots,n$:

$$(2.1.17) \Rightarrow (2.1.30).$$

$$\lambda_i = \frac{\lambda_0}{g_i}, \; i = 1,\dots,n \qquad (2.1.30)$$

$$(2.1.16) \Rightarrow (2.1.31)$$

$$\sum_{i=1}^{n} \lambda_i = 1 \qquad (2.1.31)$$

$$\sum_{i=1}^{n} \frac{\lambda_0}{g_i} = 1 \qquad (2.1.32)$$

$$\sum_{i=1}^{n} \frac{1}{g_i} = \frac{1}{\lambda_0} \qquad (2.1.33)$$

$$\lambda_0 = \frac{1}{\sum_{i=1}^{n} \frac{1}{g_i}} \qquad (2.1.34)$$

$$\lambda_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \qquad (2.1.35)$$

$$\lambda_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \; i = 1,\dots,n \qquad (2.1.36)$$

Observations:

$$x_0^{\ *} = \lambda_0^{\ *} = \left(\sum_{i=1}^{n} g_i^{\ -1} \right)^{-1} \qquad (2.1.37)$$

$$x_i^{\ *} = \lambda_i^{\ *} = g_i^{\ -1} \left(\sum_{q=1}^{n} g_q^{\ -1} \right)^{-1}, \ i = 1, \ldots, n \qquad (2.1.38)$$

The minimization problem of RED

We are interested in the solution to $\min y_0$. The objective function is formulated as $\max\left(-y_0\right)$. The frequences of the different decisions, i are y_i .

$$\max\left(-y_0\right) \tag{2.2.1}$$

s.t.

$$\sum_{i=1}^{n} y_i \geq 1 \tag{2.2.2}$$

$$y_0 \geq g_i y_i \text{ , } i = 1,\ldots,n \tag{2.2.3}$$

$$y_i \geq 0 \text{ , } i = 1,\ldots,n \tag{2.2.4}$$

Proof that $y_0^* > 0$

$$(2.2.2) \Rightarrow (2.2.5).$$

$$\exists i\big|_{1 \le i \le n,\ y_i > 0} \qquad (2.2.5)$$

$$g_i > 0\ ,\ i = 1,\ldots,n \qquad (2.2.6)$$

$$(2.2.3) \wedge (2.2.5) \not\Rightarrow {}^{\wedge} (2.2.6) \Rightarrow (2.2.7).$$

$$y_0^* \ge y_0 > 0 \qquad (2.2.7)$$

Let μ_i denote dual variables. The following Lagrange function is defined for RED:

$$L_2 = -y_0 + \mu_0\left(\sum_{i=1}^{n} y_i - 1\right) + \sum_{i=1}^{n} \mu_i\left(y_0 - g_i y_i\right) \qquad (2.2.8)$$

These derivatives will be needed in the analysis:

$$\frac{dL_2}{d\mu_0} = \sum_{i=1}^{n} y_i - 1 \ge 0 \qquad (2.2.9)$$

$$\frac{dL_2}{d\mu_i} = y_0 - g_i y_i \ge 0\ ,\ i = 1,\ldots,n \qquad (2.2.10)$$

$$\frac{dL_2}{dy_0} = -1 + \sum_{i=1}^{n} \mu_i \le 0 \qquad (2.2.11)$$

$$\frac{dL_2}{dy_i} = \mu_0 - \mu_i g_i \le 0\ ,\ i = 1,\ldots,n \qquad (2.2.12)$$

Proof that $y_i^* > 0\ ,\ i = 0,\ldots,n$

According to (2.2.1), we want to maximize $-y_0$, which implies that we minimize y_0 .

$$(2.2.2) \Rightarrow \sum_{i=1}^{n} y_i \geq 1$$

$$(2.2.4) \Rightarrow y_i \geq 0\ ,\ i = 1,\ldots,n$$

Let us start from an infeasible point, origo, and move to a feasible point in the way that keeps y_0 as low as possible. Initially, let $(y_1,\ldots,y_n) = (0,\ldots,0)$. According to (2.2.2), this point is not feasible.

$$(2.2.3) \Rightarrow \min y_0\Big|_{y_i=0\ ,\ i=1,\ldots,n} = 0\ .$$

Now, we have to move away from the infeasible point $(y_1,\ldots,y_n)=(0,\ldots,0)$. We have to reach a point that satisfies $\sum_{i=1}^{n} y_i \geq 1$ without increasing y_0 more than necessary. To find a point that satisfies (2.2.2), we have to increase the value of at least one of the $y_i\big|_{i\in\{1,\ldots,n\}}$. Select one arbitrary index $k\big|_{1\leq k\leq n}$. To simplify the exposition, we let $k=1$. According to (2.2.3): If we increase y_1 by dy_1 , $\min y_0$ increases by $g_1 dy_1$, as long as $dy_i = 0$, $i=2,\ldots,n$. Hence, $dy_0 = g_1 dy_1$. Let $z = dy_0 = g_1 dy_1$.

However, when $dy_1 > 0$, we may also partly increase y_i, $i = 2,...,n$ without increasing dy_0 above z. This follows from (2.2.3) and (2.2.10). Since we want to satisfy $\sum_{i=1}^{n} y_i \geq 1$, we want to increase y_i, $i = 2,...,n$ as much as possible, without increasing dy_0 above z. Hence, we select:

$$g_i dy_i = z = g_1 dy_1 \ , i = 2,...,n \tag{2.2.13}$$

$$dy_i = \frac{g_1}{g_i} dy_1 \ , i = 2,...,n \tag{2.2.14}$$

$$(dy_1 > 0) \wedge (g_i > 0 \ , i = 1,...,n) \Rightarrow dy_i > 0 \ , i = 2,...,n \tag{2.2.15}$$

Since we started in origo, we have

$$y_i = dy_i + 0 > 0 \ , i = 1,...,n \tag{2.2.16}$$

We already know that $y_0^* \geq y_0 > 0$. Hence,.

$$y_i^* > 0 \ , i = 0,...,n \tag{2.2.17}$$

Observation: The following direct method can be used to solve the optimization problem of RED.

First, remember that $y_0^* = dy_0^* + 0 = z$. We may directly determine the optimal values of $y_i^* > 0$, $i = 0,\dots,n$ without using the Lagrange function and KKT conditions, in this way:

$$\sum_{i=1}^{n} y_i = \left((dy_1 + 0) + (dy_2 + 0)\ldots + (dy_n + 0)\right) = 1 \qquad (2.2.18)$$

$$\sum_{i=1}^{n} y_i = (y_1 + y_2 + \ldots + y_n) = 1 \qquad (2.2.19)$$

$$\sum_{i=1}^{n} y_i = \left(\frac{z}{g_1} + \left(\frac{g_1}{g_2}\frac{z}{g_1}\right) + \ldots + \left(\frac{g_1}{g_n}\frac{z}{g_1}\right)\right) = 1 \qquad (2.2.20)$$

$$\sum_{i=1}^{n} y_i = \left(\frac{z}{g_1} + \frac{z}{g_2} + \ldots + \frac{z}{g_n}\right) = 1 \qquad (2.2.21)$$

$$\sum_{i=1}^{n} y_i = \left(\frac{1}{g_1} + \frac{1}{g_2} + \ldots + \frac{1}{g_n}\right) = \frac{1}{z} \qquad (2.2.22)$$

$$\sum_{i=1}^{n} g_i^{-1} = \frac{1}{z} \qquad (2.2.23)$$

$$y_0^* = z = \left(\sum_{i=1}^{n} g_i^{-1}\right)^{-1} \qquad (2.2.24)$$

$$y_i^* = g_i^{-1} y_0^* = g_i^{-1}\left(\sum_{q=1}^{n} g_q^{-1}\right)^{-1}, \; i = 1,\dots,n \qquad (2.2.25)$$

$$y_0^* = z = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \tag{2.2.24}$$

$$y_i^* = g_i^{-1} y_0^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, i = 1, \ldots, n \tag{2.2.25}$$

Proof that $\mu_i^*, i=0,\ldots,n$ can be solved via a linear equation system and that $\mu_i^*>0, i=0,\ldots,n$.

Since $y_i^*>0$, $i=0,\ldots,n$, we may determine that $\mu_i^*>0$, $i=0,\ldots,n$ via a linear equation system.

$$\left(y_i\frac{dL_2}{dy_i}=0\,,\,i=0,\ldots,n\right)\wedge\left(y_i>0\,,\,i=0,\ldots,n\right)\Rightarrow\left(\frac{dL_2}{dy_i}=0\,,\,i=0,\ldots,n\right)$$

$$\frac{dL_2}{dy_0}=-1+\sum_{q=1}^{n}\mu_q=0 \qquad (2.2.26)$$

$$\frac{dL_2}{dy_i}=\mu_0-\mu_i g_i=0\,,\,i=1,\ldots,n \qquad (2.2.27)$$

$$(2.2.26)\Rightarrow\ \exists i\big|_{1\le i\le n,\ \mu_i>0} \qquad (2.2.28)$$

$$\left(g_i>0\,,\,i=1,\ldots,n\right)\wedge(2.2.27)\ \wedge\ (2.2.28)\ \Rightarrow\mu_0>0 \qquad (2.2.29)$$

$$\left(g_i>0\,,\,i=1,\ldots,n\right)\wedge(2.2.27)\ \wedge\ (2.2.29)\ \Rightarrow\left(\mu_i>0\,,\,i=1,\ldots,n\right)$$

$$(2.2.30)$$

$$(2.2.29)\ \wedge\ (2.2.30)\ \Rightarrow\left(\mu_i>0\,,\,i=0,\ldots,n\right) \qquad (2.2.31)$$

Proof that $y_i^*, i=0,\ldots,n$ can be solved via a linear equation system and that $y_i^*>0, i=0,\ldots,n$.

Since $\mu_i^*>0, i=0,\ldots,n$, we may determine that $y_i^*>0, i=0,\ldots,n$ via a linear equation system.

$$\left(\mu_i\frac{dL_2}{d\mu_i}=0, i=0,\ldots,n\right)\wedge\left(\mu_i>0, i=0,\ldots,n\right)\Rightarrow\left(\frac{dL_2}{d\mu_i}=0, i=0,\ldots,n\right)$$

$$\frac{dL_2}{d\mu_0}=\sum_{q=1}^{n}y_q-1=0 \qquad (2.2.32)$$

$$\frac{dL_2}{d\mu_i}=y_0-g_iy_i=0, i=1,\ldots,n \qquad (2.2.33)$$

$$(2.2.32)\Rightarrow\exists i\big|_{1\le i\le n,\, y_i>0} \qquad (2.2.34)$$

$$\left(g_i>0, i=1,\ldots,n\right)\wedge(2.2.33)\Rightarrow y_0>0 \qquad (2.2.35)$$

$$\left(g_i>0, i=1,\ldots,n\right)\wedge(2.2.35)\Rightarrow\left(y_i>0, i=1,\ldots,n\right) \qquad (2.2.36)$$

$$(2.2.35)\wedge(2.2.36)\Rightarrow\left(y_i>0, i=0,\ldots,n\right) \qquad (2.2.37)$$

Determination of explicit equations that give all values: y_i^* , $i = 0,\ldots,n$:

(2.2.33) $\Rightarrow$ (2.2.38).

$$y_i = \frac{y_0}{g_i}, i = 1,\ldots,n \tag{2.2.38}$$

(2.2.32) $\Rightarrow$ (2.2.39).

$$\sum_{i=1}^{n} y_i = 1 \tag{2.2.39}$$

$$\sum_{i=1}^{n} \frac{y_0}{g_i} = 1 \tag{2.2.40}$$

$$\sum_{i=1}^{n} \frac{1}{g_i} = \frac{1}{y_0} \tag{2.2.41}$$

$$y_0 = \frac{1}{\sum_{i=1}^{n} \frac{1}{g_i}} \qquad (2.2.42)$$

$$y_0^{\;*} = \left(\sum_{i=1}^{n} g_i^{\;-1} \right)^{-1} \qquad (2.2.43)$$

$$y_i^{\;*} = g_i^{\;-1} \left(\sum_{q=1}^{n} g_q^{\;-1} \right)^{-1}, \; i = 1, \ldots, n \qquad (2.2.44)$$

Determination of explicit equations that give all values: μ_i^* , $i = 0,\dots,n$:

(2.2.27) $\Rightarrow$ (2.2.45).

$$\mu_i = \frac{\mu_0}{g_i}, \; i = 1,\dots,n \tag{2.2.45}$$

(2.2.26) $\Rightarrow$ (2.2.46)

$$\sum_{i=1}^{n} \mu_i = 1 \tag{2.2.46}$$

$$\sum_{i=1}^{n} \frac{\mu_0}{g_i} = 1 \tag{2.2.47}$$

$$\sum_{i=1}^{n} \frac{1}{g_i} = \frac{1}{\mu_0} \tag{2.2.48}$$

$$\mu_0 = \frac{1}{\sum_{i=1}^{n} \frac{1}{g_i}} \tag{2.2.49}$$

$$\mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \tag{2.2.50}$$

$$\mu_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \; i = 1,\dots,n \tag{2.2.51}$$

Observations:

$$y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \qquad (2.2.52)$$

$$y_i^* = \mu_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, i = 1, \ldots, n \qquad (2.2.53)$$


New General Results

Generalized Observations:

$$x_0^* = \lambda_0^* = y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \qquad (2.2.54)$$

$$x_i^* = \lambda_i^* = y_i^* = \mu_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, i = 1, \ldots, n \quad (2.2.55)$$

Sensitivity analyses

First, the sensitivity analyses will concern these variables: $x_0^* = \lambda_0^* = y_0^* = \mu_0^*$. How do these variables change under the influence of changing elements in the game matrix?

Observation: $x_0^* = \lambda_0^* = y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1}$

Proof that $\frac{dx_0^*}{dg_i} > 0 \wedge \frac{d^2x_0^*}{dg_i^2} < 0$.

$$x_0^* = \left(\sum_{i=1}^{n} g_i^{-1}\right)^{-1} \tag{2.3.1}$$

$$\frac{dx_0^*}{dg_i} = (-1)\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-2}\left(-g_i^{-2}\right) \tag{2.3.2}$$

$$\frac{dx_0^*}{dg_i} = g_i^{-2}\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-2} > 0 \tag{2.3.3}$$

$$\frac{d^2x_0^*}{dg_i^2} = -2g_i^{-3}\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-2} + g_i^{-2}(-2)\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-3}(-1)g_i^{-2} \tag{2.3.4}$$

$$\frac{d^2x_0^*}{dg_i^2} = -2g_i^{-3}\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-2}\left(1 - g_i^{-1}\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-1}\right) \tag{2.3.5}$$

$$\frac{d^2x_0^*}{dg_i^2} = -2g_i^{-1}\left(x_i^*\right)^2\left(1 - x_i^*\right) \tag{2.3.6}$$

$$\left(0 < x_i^* < 1\right) \wedge \left(g_i > 0\right) \Rightarrow \frac{d^2x_0^*}{dg_i^2} < 0 \tag{2.3.7}$$

New General Results

Observation: x_0^* is a strictly increasing and strictly concave function of each g_i. From the Jensen inequality, it follows that increasing risk in g_i will reduce the expected value of x_0^*. Compare Figure 1.

Second, the sensitivity analyses will concern these variables: $x_i^* = \lambda_i^* = y_i^* = \mu_i^*$, $i = 1,\ldots,n$. How do these variables change under the influence of changing elements in the game matrix?

Observation: $x_i^* = \lambda_i^* = y_i^* = \mu_i^* = g_i^{-1}\left(\sum_{q=1}^{n} g_q^{-1}\right)^{-1}$, $i = 1,\ldots,n$

Proof that $\frac{dx_i^*}{dg_i} < 0 \wedge \frac{d^2 x_i^*}{dg_i^2} > 0$, $i \in \{1, \ldots, n\}$.

$$x_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \; i = 1, \ldots, n \qquad (2.3.8)$$

$$\frac{dx_i^*}{dg_i} = -g_i^{-2} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1} + g_i^{-1}(-1) \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-2} \left(-g_q^{-2} \right) \qquad (2.3.9)$$

$$\frac{dx_i^*}{dg_i} = g_i^{-2} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1} \left(-1 + g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1} \right) \qquad (2.3.10)$$

$$\frac{dx_i^*}{dg_i} = g_i^{-1} x_i^* \left(-1 + x_i^* \right) \qquad (2.3.11)$$

$$(g_i > 0) \wedge \left(0 < x_i^* < 1\right) \Rightarrow \frac{dx_i^*}{dg_i} < 0 \qquad (2.3.12)$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2} x_i^* \left(x_i^* - 1\right) + g_i^{-1}\left(g_i^{-1} x_i^* \left(x_i^* - 1\right)\right)\left(x_i^* - 1\right) + g_i^{-1} x_i^* g_i^{-1} x_i^* \left(x_i^* - 1\right)$$

$$(2.3.13)$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2}\left(x_i^*\left(x_i^* - 1\right) - \left(x_i^*\left(x_i^* - 1\right)\right)\left(x_i^* - 1\right) - x_i^* x_i^*\left(x_i^* - 1\right)\right)$$

$$(2.3.14)$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2}\left(\left(x_i^*\right)^2 - x_i^* - x_i^*\left(\left(x_i^*\right)^2 - 2x_i^* + 1\right) - \left(x_i^*\right)^2\left(x_i^* - 1\right)\right)$$

$$(2.3.15)$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2}\left(\left(x_i^*\right)^2 - x_i^* - \left(x_i^*\right)^3 + 2\left(x_i^*\right)^2 - x_i^* - \left(x_i^*\right)^3 + \left(x_i^*\right)^2\right) \tag{2.3.16}$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2}\left(-2\left(x_i^*\right)^3 + 4\left(x_i^*\right)^2 - 2x_i^*\right) \tag{2.3.17}$$

$$\frac{d^2 x_i^*}{dg_i^2} = 2g_i^{-2}x_i^*\left(\left(x_i^*\right)^2 - 2x_i^* + 1\right) \tag{2.3.18}$$

$$\frac{d^2 x_i^*}{dg_i^2} = 2g_i^{-2}x_i^*\left(x_i^* - 1\right)^2 \tag{2.3.19}$$

$$\left(g_i \neq 0\right) \wedge \left(0 < x_i^* < 1\right) \Rightarrow \frac{d^2 x_i^*}{dg_i^2} > 0 \tag{2.3.20}$$


New General Results

Observation: x_i^* is a strictly decreasing and strictly convex function of g_i . From the Jensen inequality, it follows that increasing risk in g_i will increase the expected value of x_i^* . Compare Figure 2.

Proof that $\frac{dx_k^*}{dg_i} > 0 \wedge \frac{d^2 x_k^*}{dg_i^2} < 0$, $i \in \{1,\ldots,n\}, k \in \{1,\ldots,n\}, i \neq k$.

$$x_k^* = g_k^{-1} \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \qquad (2.3.21)$$

$$\frac{dx_k^*}{dg_{i|i \neq k}} = g_k^{-1} (-1) \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-2} \left(-g_i^{-2} \right) \qquad (2.3.22)$$

$$\frac{dx_k^*}{dg_{i|i \neq k}} = g_k^{-1} g_i^{-2} \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-2} \qquad (2.3.23)$$

$$\left(g_m > 0 \text{ , } m = 1\ldots,n) \right) \Rightarrow \frac{dx_k^*}{dg_{i|i \neq k}} > 0 \qquad (2.3.24)$$

$$\frac{d^2 x_k^{\ *}}{dg_{i|i\neq k}^{\ \ 2}} = g_k^{\ -1}\left(-2g_i^{\ -3}\left(\sum_{i=1}^{n} g_i^{\ -1}\right)^{-2} + g_i^{\ -2}(-2)\left(\sum_{i=1}^{n} g_i^{\ -1}\right)^{-3}\left(-g_i^{\ -2}\right)\right)$$

$$(2.3.25)$$

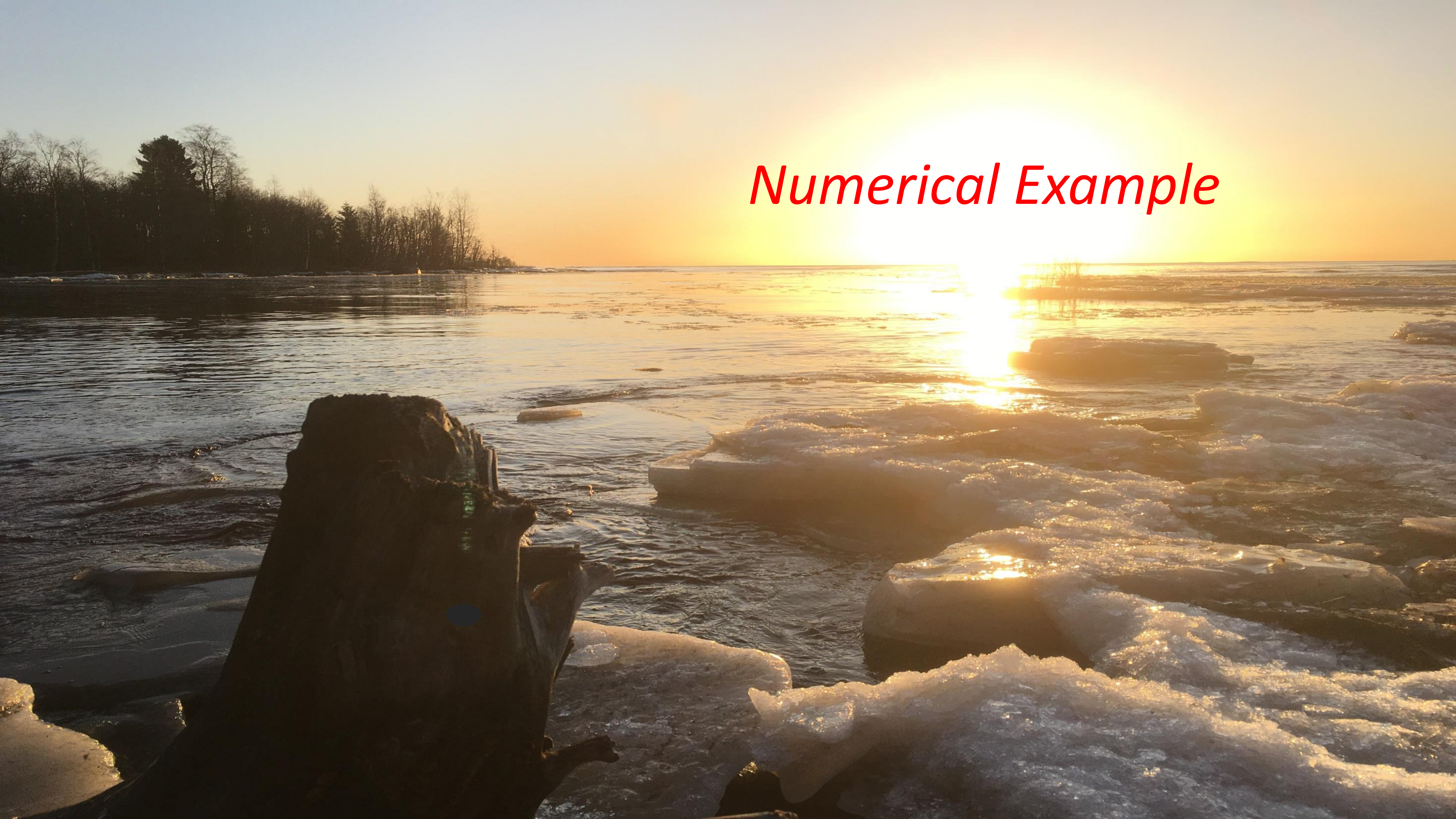
$$\frac{d^2 x_k^{\ *}}{dg_{i|i\neq k}^{\ \ 2}} = 2g_k^{\ -1} g_i^{\ -3}\left(\sum_{i=1}^{n} g_i^{\ -1}\right)^{-2}\left(\left(g_i^{\ -1}\right)\left(\sum_{i=1}^{n} g_i^{\ -1}\right)^{-1} - 1\right) \quad (2.3.26)$$


New General Results

$$\frac{d^2 x_k^*}{d g_{i|i \neq k}^2} = 2 g_k^{-1} g_i^{-1} \left(x_i^*\right)^2 \left(x_i^* - 1\right) \qquad (2.3.27)$$

$$\left(g_m > 0\,,\, m = 1,\ldots,n\right) \wedge \left(0 < x_i^* < 1\right) \Rightarrow \frac{d^2 x_k^*}{d g_{i|i \neq k}^2} < 0 \qquad (2.3.28)$$

Observation: x_k^* is a strictly increasing and strictly concave function of g_i. From the Jensen inequality, it follows that increasing risk in g_i will decrease the expected value of x_k^*. Compare Figure 3.


Numerical Example

Numerical illustration

The general definition of the following illustrating game is given in the preceeding section. Let n =2. A very detailed background and interpretation of this particular game, without the new functions and proofs, is given in Lohmander (2019).[14]

$$A = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad (3.1)$$

From (2.2.54) we know that:

$$x_0^* = \lambda_0^* = y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \qquad (3.2)$$

x_0^*, the expected reward of BLUE, is equal to y_0^*, the expected loss of RED, in case both optimize the respective strategies. Using the numerical values of the elements in A, we get:

$$x_0^* = \frac{1}{\frac{1}{2}+\frac{1}{3}} = \frac{6}{5} = 1.2 \qquad (3.3)$$

Hence, the expected value of the game is 1.2. This value is also shown in Figure 4. and Figure 5. The expected value of the game is a decreasing function of the level of risk of g_1, which is described in connection to, and illustrated in, Figure 1.

From (2.2.55) we know that:

$$x_i^* = \lambda_i^* = y_i^* = \mu_i^* = g_i^{-1}\left(\sum_{q=1}^{n} g_q^{-1}\right)^{-1}, i = 1, \ldots, n \qquad (3.4)$$

For BLUE and RED, the optimal probabilities to select different roads are equal. For BLUE, the optimal probability to select road 1 is x_1^*. Via the elements in A, we get:

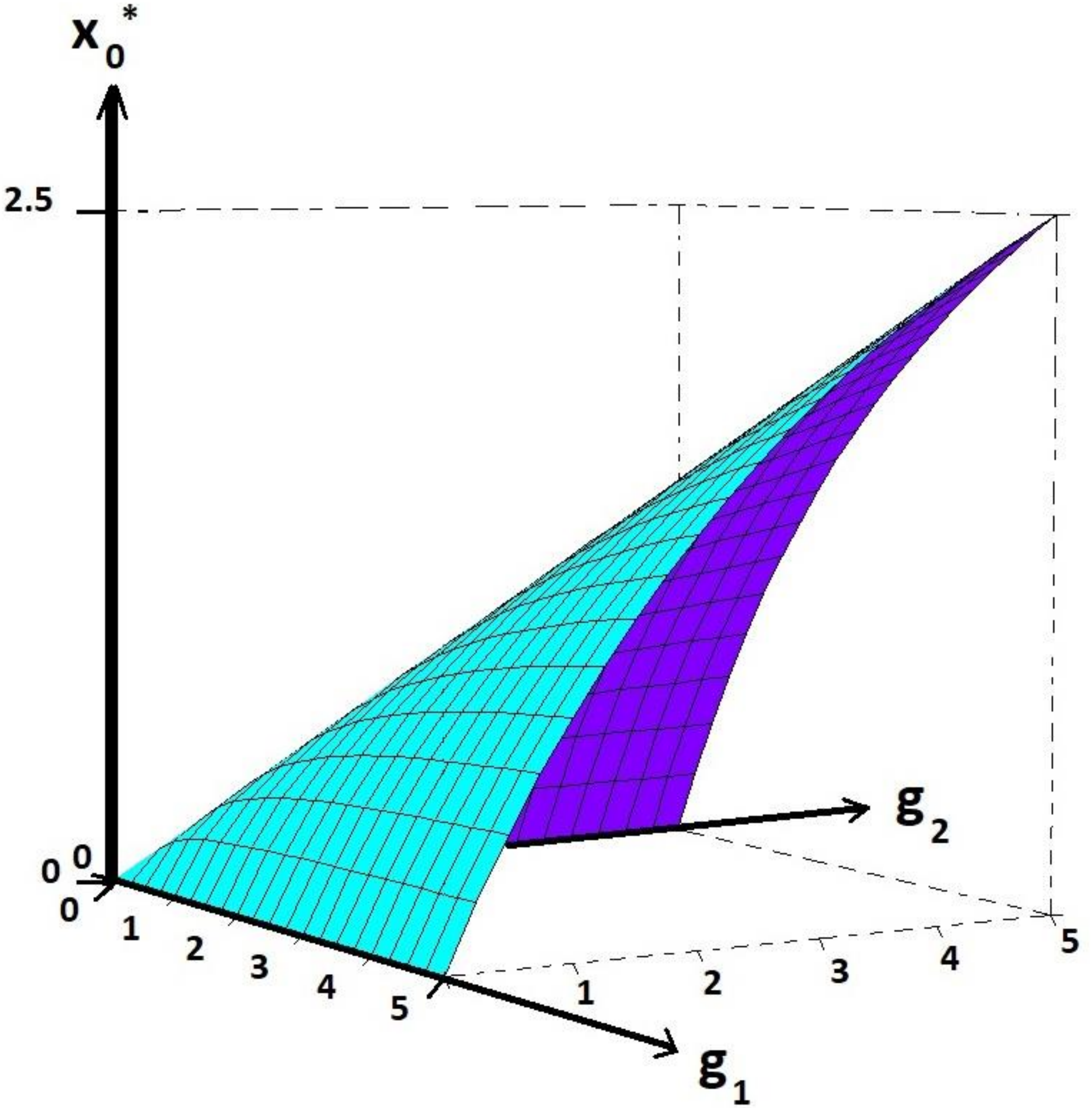
$$x_1^* = y_1^* = \left(\frac{1}{2}\right)x_0^* = 0.6 \qquad (3.5)$$

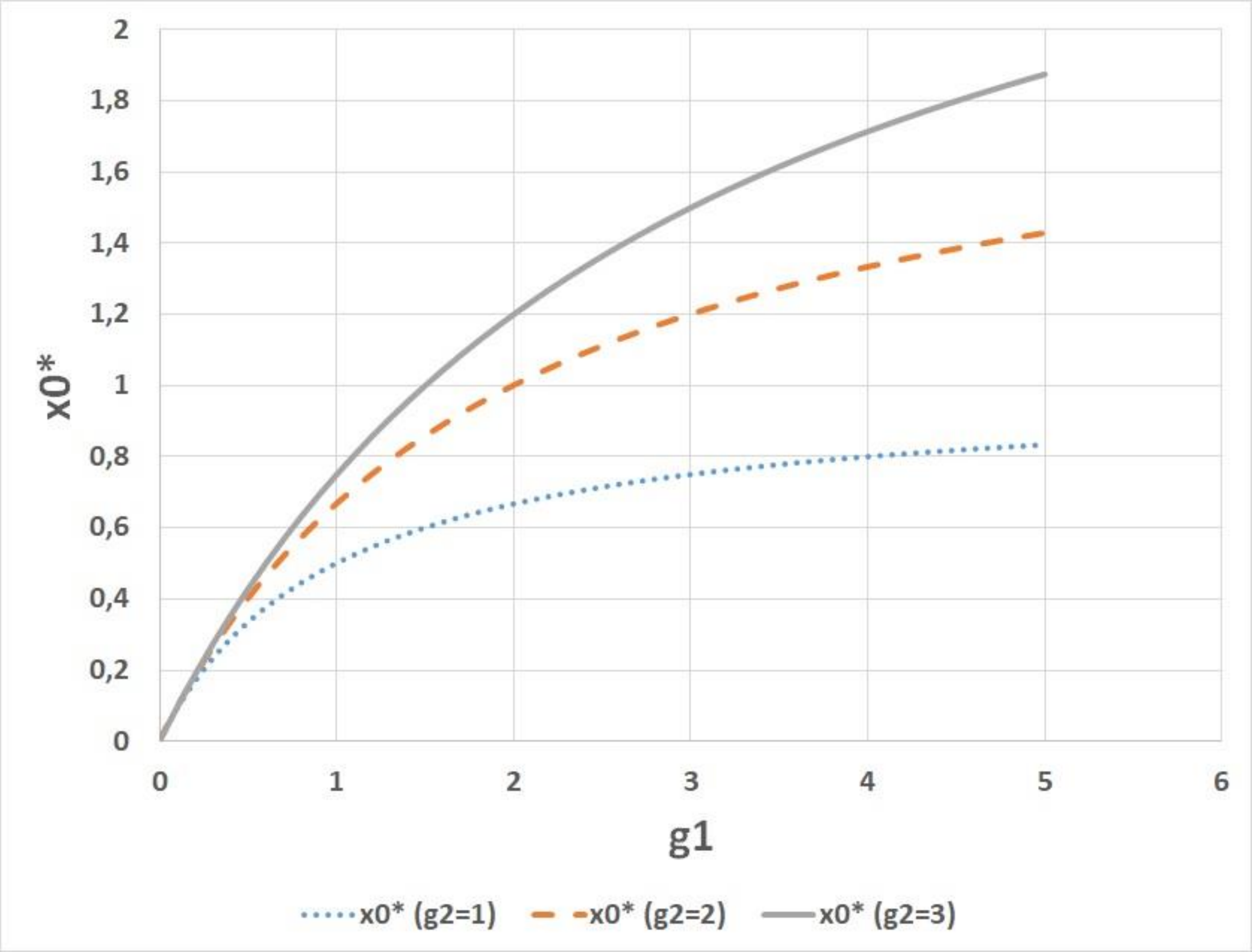
$$x_2^* = y_2^* = \left(\frac{1}{3}\right)x_0^* = 0.4 \qquad (3.6)$$

x_1^* is shown in Figures 6 & 7. In Figure 8, the optimal value is illustrated. The expected value of x_1^* is an increasing function of the level of risk in g_1, which is shown in Figure 2. For BLUE, the optimal probability to select road 2, is x_2^*. In Figure 9, we find this value is 0.4. Figure 3 illustrates that the expected value of x_2^* is a decreasing function of the level of risk in g_1.

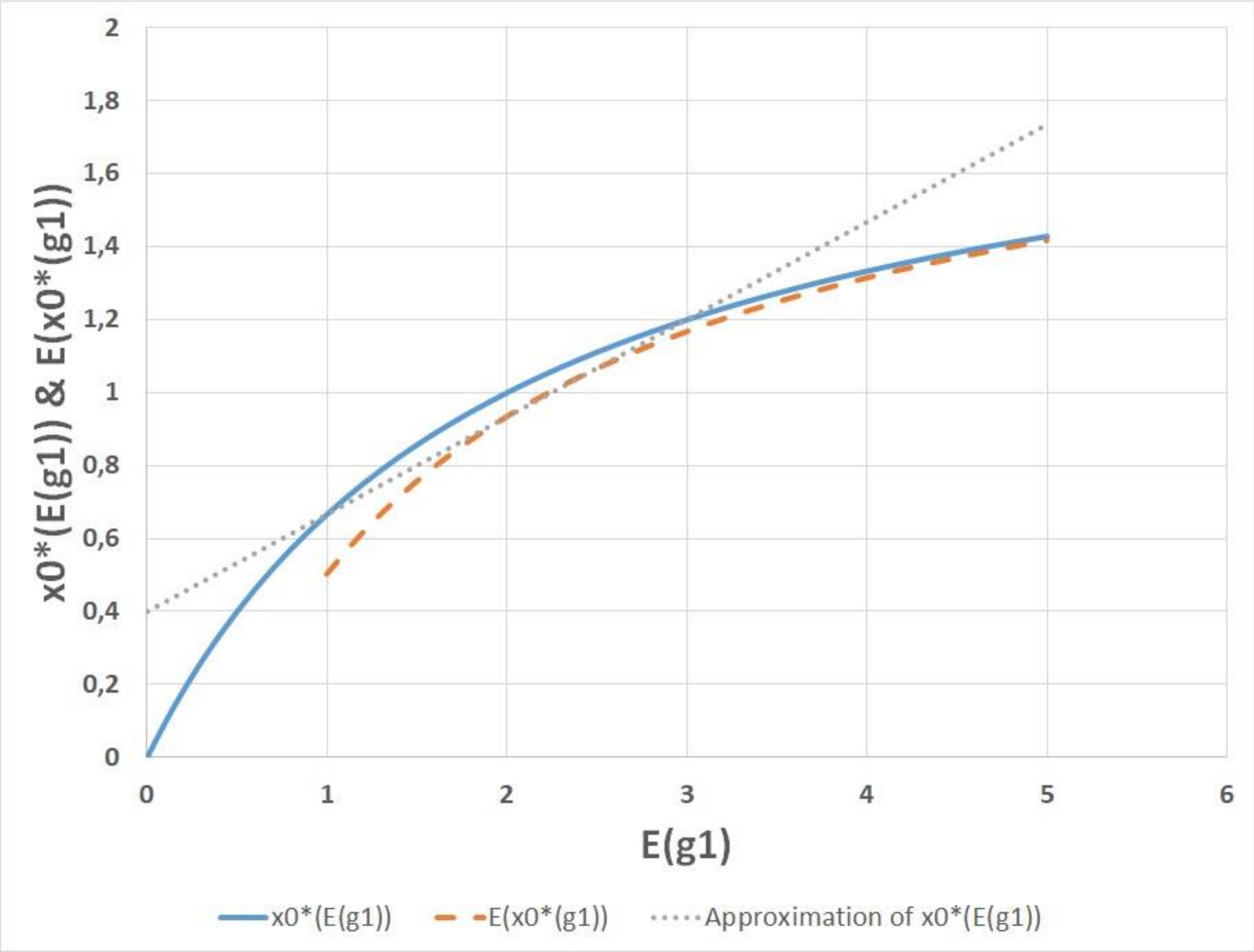
The particular results $\left(x_0^*, x_1^*, x_2^*\right)$ discussed in this in this section were also obtained by Lohmander (2019)[14] via the traditional game theory approach of linear programming.


Graphical illustrations
of optimal results

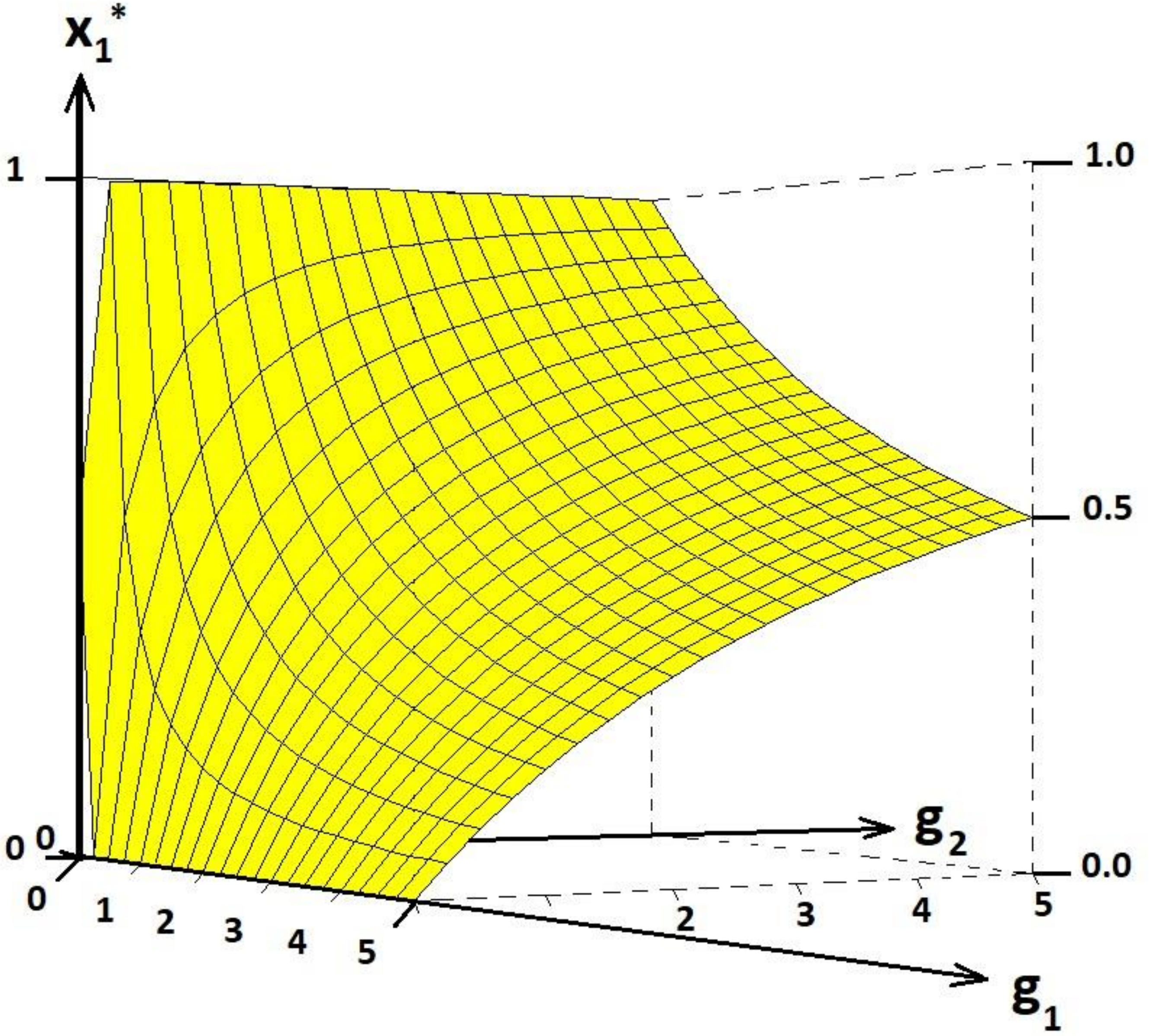

x_0^*
2.5
0
0
0
1
2
3
4
5
g_1
1
2
3
4
5
g_2

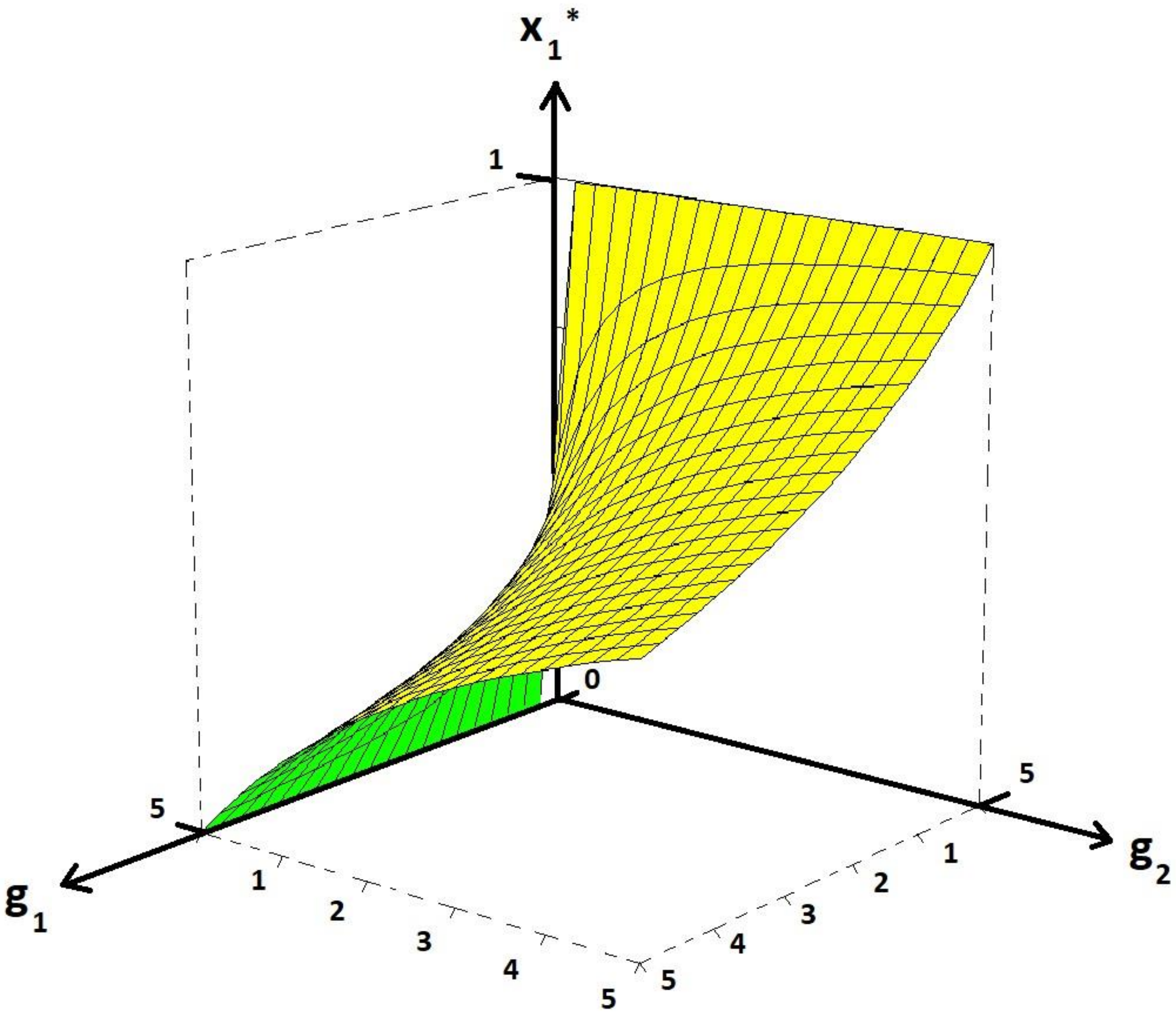


2
1,8
1,6
1,4
1,2
1
0,8
0,6
0,4
0,2
0
x0*
0
1
2
3
4
5
6
g1
x0* (g2=1)
x0* (g2=2)
x0* (g2=3)

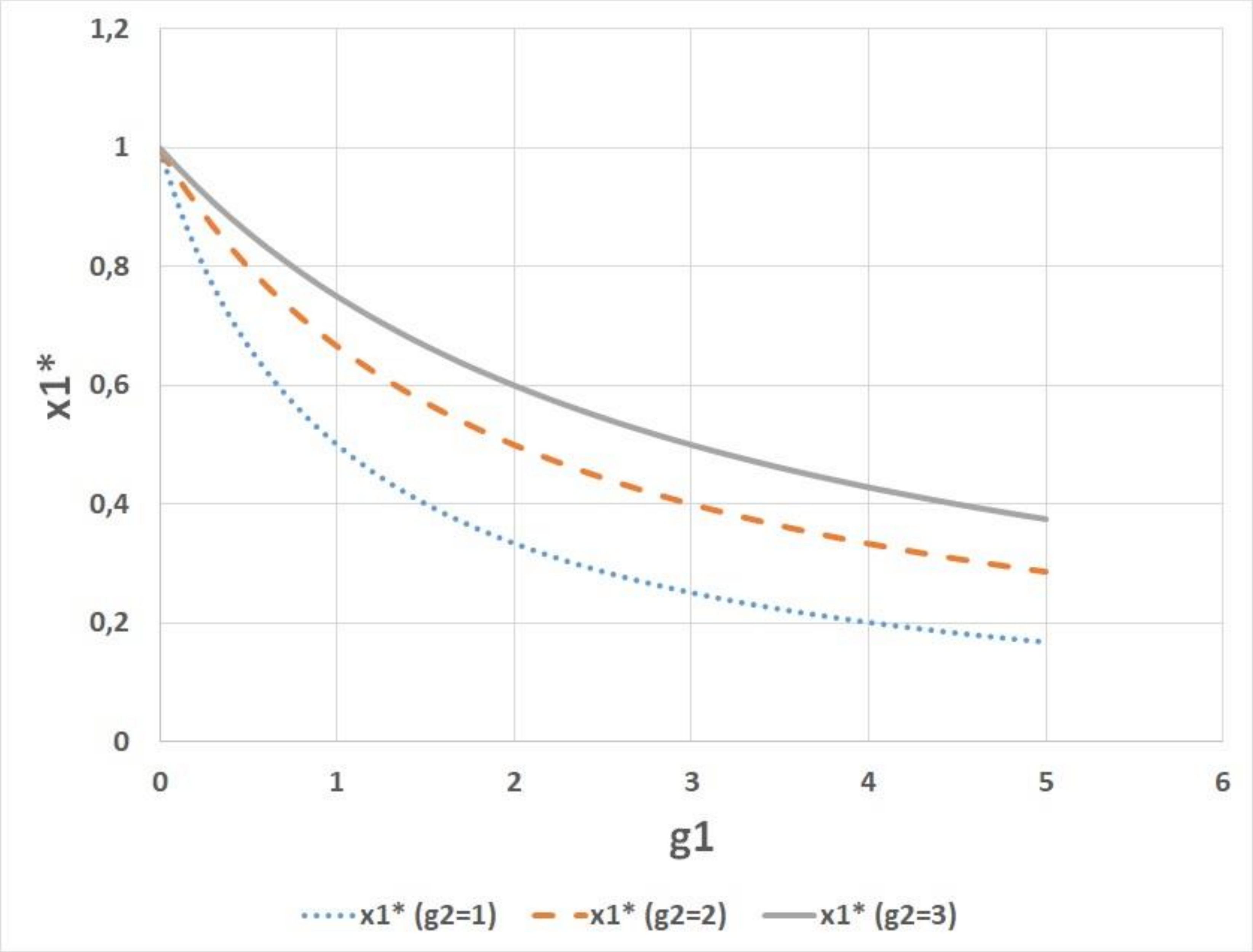


x0*(E(g1)) & E(x0*(g1))
2
1,8
1,6
1,4
1,2
1
0,8
0,6
0,4
0,2
0
0
1
2
3
4
5
6
E(g1)
x0*(E(g1))
E(x0*(g1))
Approximation of x0*(E(g1))

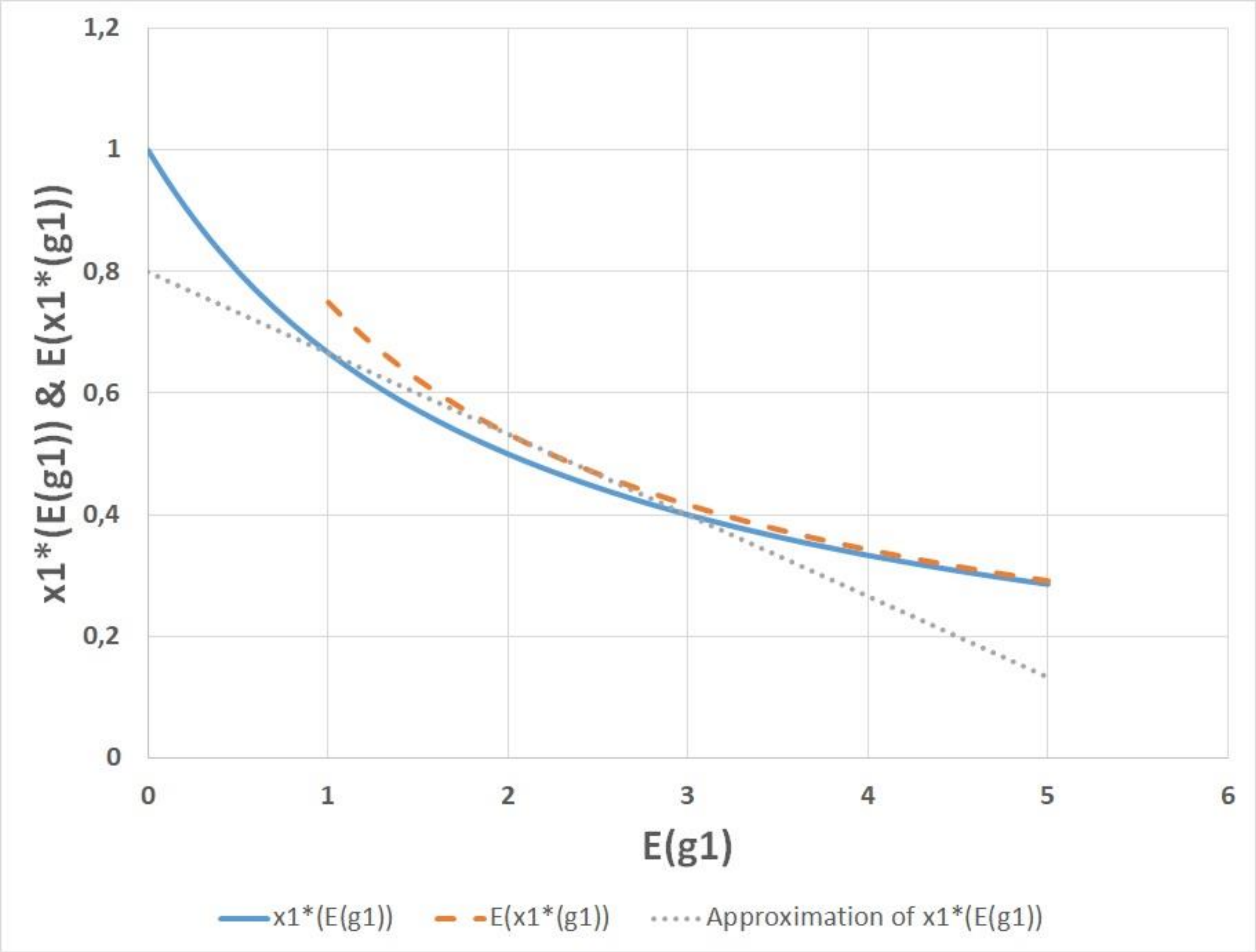

x_1^*
1
1.0
0.5
0.0
0
0
0
1
2
3
4
5
2
3
4
5
g_2
g_1



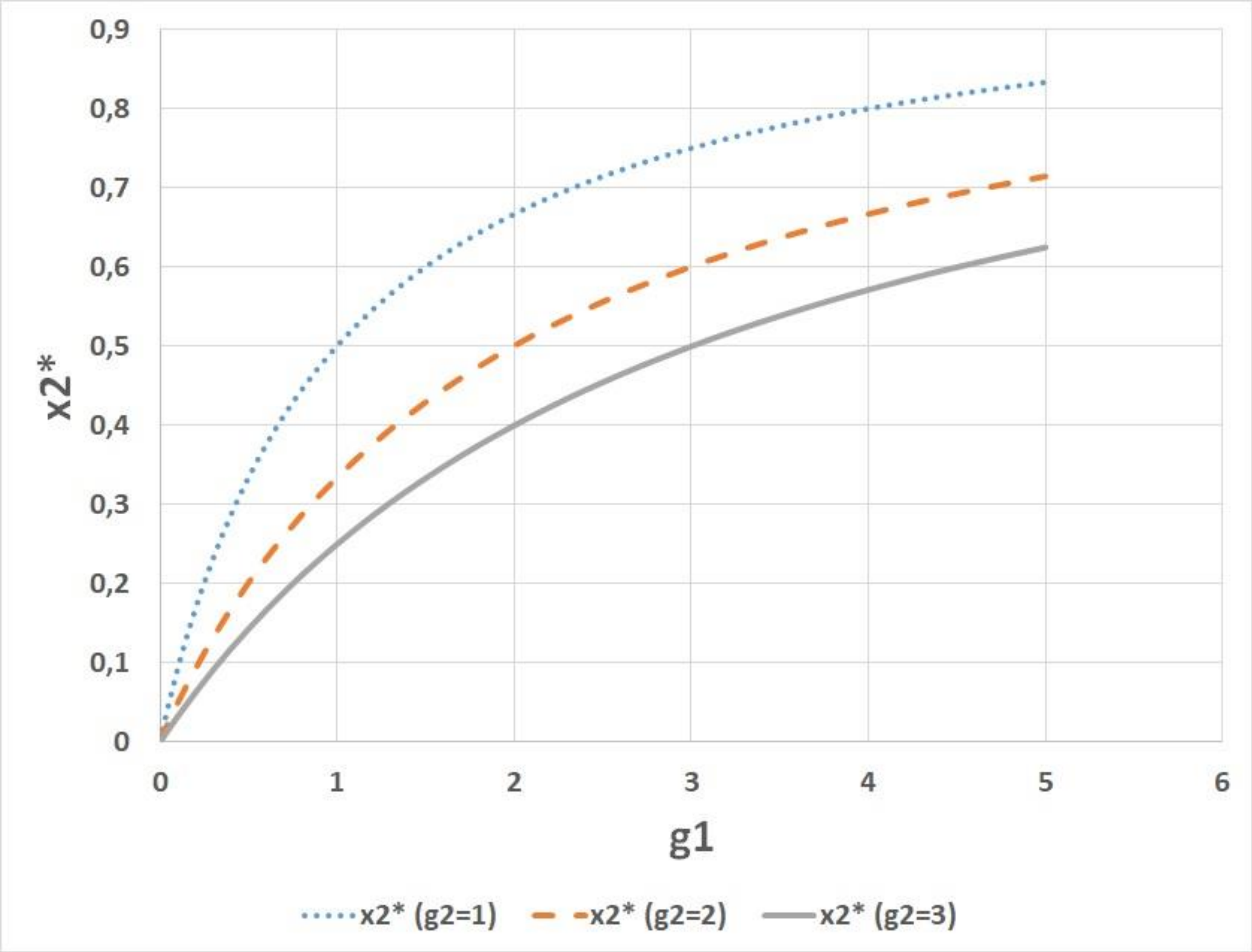
x_1^*
1
0
5
g_1
1
2
3
4
5
5
g_2
1
2
3
4
5

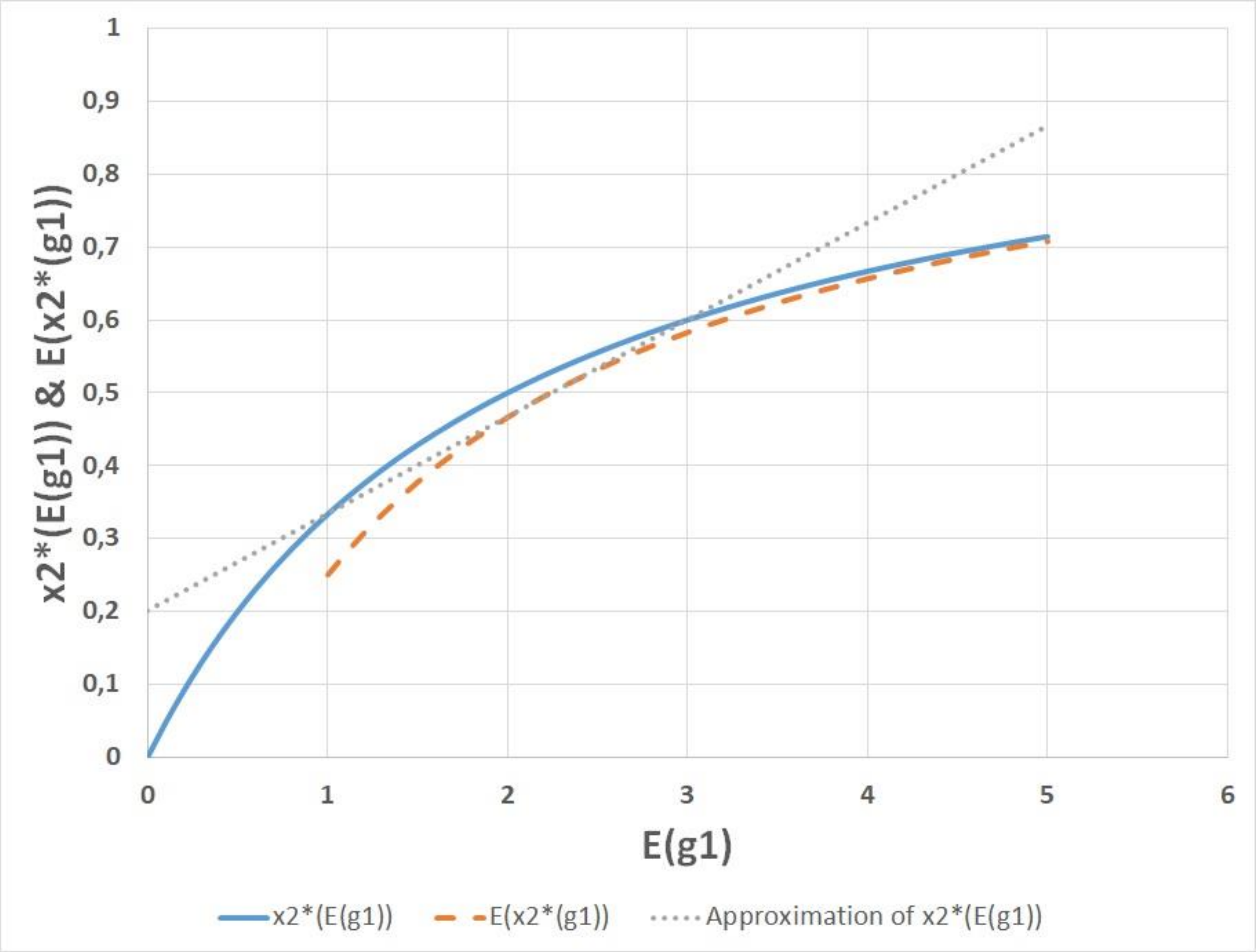


1,2
1
0,8
0,6
0,4
0,2
0
x1*
0
1
2
3
4
5
6
g1
x1* (g2=1)
x1* (g2=2)
x1* (g2=3)



1,2
1
0,8
0,6
0,4
0,2
0
0
1
2
3
4
5
6
x1*(E(g1)) & E(x1*(g1))
E(g1)
x1*(E(g1))
E(x1*(g1))
Approximation of x1*(E(g1))


0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
x2*
0
1
2
3
4
5
6
g1
x2* (g2=1)
x2* (g2=2)
x2* (g2=3)



x2*(E(g1)) & E(x2*(g1))
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
0
1
2
3
4
5
6
E(g1)
x2*(E(g1))
E(x2*(g1))
Approximation of x2*(E(g1))


On Stochastic
Dynamic Games

APPLICATIONS AND MATHEMATICAL MODELING IN OPERATIONS RESEARCH

Version 160828

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Keynote presentation at:

International Conference on Mathematics and Decision Sciences

September 12-15, 2016, Guangzhou, China

Lohmander, P., **Applications and Mathematical Modeling in Operations Research**, In: Cao BY. (ed) Fuzzy Information and Engineering and Decision. IWDS 2016. ***Advances in Intelligent Systems and Computing***, vol 646. Springer, Cham, 2018 Print ISBN 978-3-319-66513-9, Online ISBN 978-3-319-66514-6, eBook Package: Engineering

$$(15)\quad Z(t,s_{At},s_{Bt},m)=\min_{v\in V(t,s_{Bt},m)}\max_{u\in U(t,s_{At},m)}\left(\left(\begin{array}{l}\min\limits_{y\in Y(t,s_{Bt},u,v,m)}\max\limits_{x\in X(t,s_{At},u,v,m)} Q(x,y;u,v,t,s_{At},s_{Bt},m)\\ \qquad s.t.\\ \qquad F_{1,f_1}(x,y)\le 0\ \forall f_1\\ \qquad F_{2,f_2}(x,y)\ge 0\forall f_2\\ \qquad F_{3,f_3}(x,y)=0\forall f_3\end{array}\right)+\sum_n \tau(n|m)Z(t+1,s_{A(t+1)}(s_{At},t,m,v,u),s_{B(t+1)}(s_{Bt},t,m,v,u),n)\right)$$

$$\forall(t,s_{At},s_{Bt},m)\big|\left(0\le t\le T\right)$$

$$(16)\quad Z(T+1,s_{At},s_{Bt},m)=0\quad\forall(s_{At},s_{Bt},m)$$

Stochastic dynamic games with arbitrary functions, with and without mixed strategies

$$V(x_t, y_t) = \max_{GS_{1_t}, CA_{1_t}} \min_{GS_{2_t}, CA_{2_t}} \left\{ R_t(\bullet) + d \sum_{x_{t+1}} \sum_{y_{t+1}} \tau(x_{t+1}, y_{t+1} | \bullet) V(x_{t+1}, y_{t+1}) \right\} \qquad \forall t \Big|_{t<T}$$

$$(GS_{1_t}, CA_{1_t}) \in A_1(x_t)$$

$$(GS_{2_t}, CA_{2_t}) \in A_2(y_t)$$

$$t \in \{0,1,...,T-1\}$$

$$x_t \in \{0,1,...,N_x\} \forall t$$

$$y_t \in \{0,1,...,N_y\} \forall t$$

Lohmander, P., A Stochastic Differential (Difference) Game Model With an LP Subroutine for Mixed and Pure Strategy Optimization, INFORMS International Meeting 2007, Puerto Rico, 2007 http://www.Lohmander.com/SDG.ppt

General Conclusions

General Conclusions:

Game theory is necessary in order to understand and handle relevant decision problems.

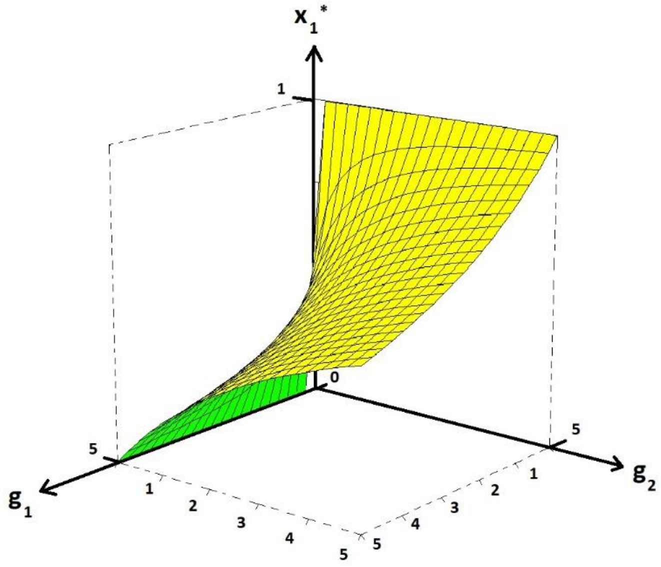
Game theory contains an enormous number of alternative specifications.

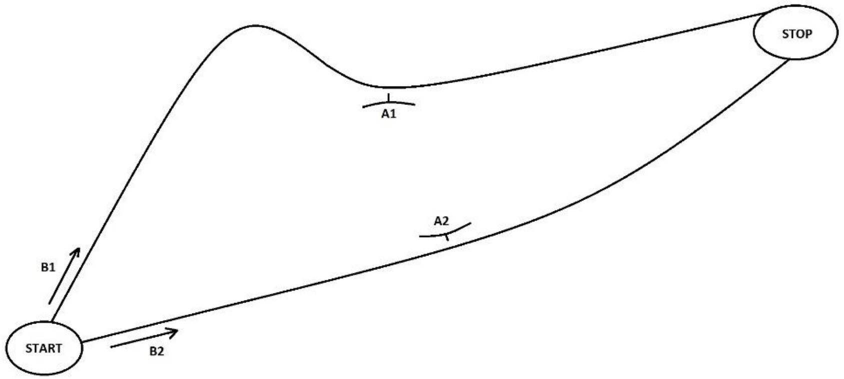
It is essential that the most relevant approach is defined, analyzed and used.

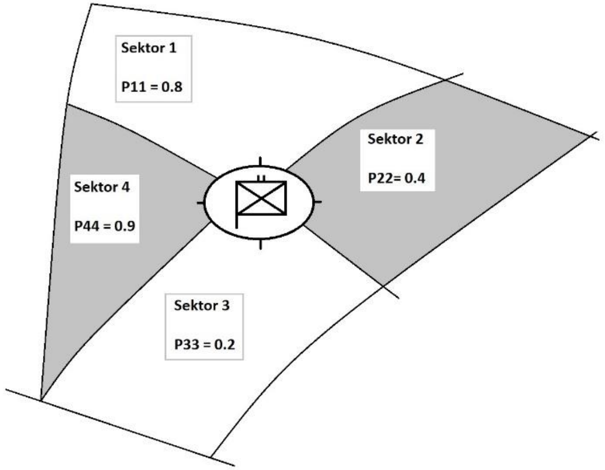
I hope that we can cooperate in this field in the future.

Peter Lohmander

RECENT ADVANCES IN GENERAL GAME THEORY AND APPLICATIONS







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2020-01-16 13.00 – 14.30