Optimal Strategies and Tactical Decisions for Army Brigades

Peter Lohmander
Peter Lohmander Optimal Solutions, Hoppets Grand 6, SE 903 34 Umea, Sweden.
e-mail: Peter@Lohmander.com

ABSTRACT

Strategies and tactical decisions, of an army brigade, are optimized. The objective function is based on all costs of the preparations and of the war. In the first section, the optimal depth of defense, Depth, is determined. Simultaneously, the optimal levels of field work for attack, FWA, and for protection, FWP, are derived. FWA includes defense preparations with vehicle mines, antipersonnel mines, destruction of bridges and preparations of assaults and counter attacks. Construction of fortifications and tunnels are parts of FWP. The initial number of defending soldiers, x_0, is simultaneously calculated. A differential equation system model of an extended Lanchester type is constructed and solved, with attrition parameters that are functions of Depth, FWA and FWP. The differential equation system model is used as a sub routine in a new defense optimization model. The time before the war ends, T, and the number of unharmed defending soldiers, x_T, are simultaneously determined as functions of the parameters and optimal decisions. The optimal objective function value is a decreasing function of the price of the land and of the price of the soldiers. In optimum: Depth and x_0 are decreasing functions of the respective prices; FWP is an increasing function of the prices of land and soldiers, and FWA is an increasing function of the price of land. In the second section, the activities within the defending brigade, BLUE, and the attacking force, RED, are analyzed in detail. Artillery battalions, mechanized battalions and cavalry ranger battalions cooperate with each other and with other kinds of units. A controlled discrete time nonlinear difference equation system in six dimensions is constructed and numerically solved. This model system represents the war that takes place within the area defended by the BLUE brigade. The model describes the dynamic developments of the BLUE and RED resources. The model optimizes the degree of cooperation and selection of targets by different kinds of battalions. The attrition levels are nonlinear functions of the amounts of time that different kinds of battalions spend in different ways in this cooperation. For instance, a cavalry ranger battalion may search and find targets for an artillery battalion, and/or attack enemy cavalry units or other enemies. Each battalion has limited total time available. The model is used to derive the dynamic development of the resources of BLUE and RED under different parameter assumptions, representing dynamic unit deployments and other conditions. The model can with marginal adjustments also be used to determine the dynamic game outcomes when BLUE and RED sequentially adapt the decisions to the changes made by the other sides. As an illustration, the optimal use of cavalry ranger battalion time as a function of the cost per unit of the remaining enemy artillery battalions at a future point in time, is determined.

Keywords: Military optimization, strategy, tactics, coordination, army, brigade, battalion.

1. Introduction

The optimal protection of a nation against possible attacks, or invasions, from hostile enemies, is a very important issue in most countries. The conditions of relevance to rational defense, such as total area of the country, the population density, the climate and the terrain, can differ very much between different countries. The ambition in this paper is to investigate this optimal defense problem in a general way, that is possible to apply in different regions with very different conditions. In the future, it will hopefully be possible to generalize the approach to cover all kinds of military units. This first version will however be constrained to the army, based on artillery, cavalry rangers, mechanized battalions, engineers and more.

1.1. Military analysis without formal methods

The literature contains detailed information on historical wars and military forces. Two good examples are Jörgensen (2007), an excellent example of the many books that describe large numbers of historical battles, and Gyllenhaal (2009), who focuses on elite forces in the Nordic countries.

One example of books that intends to teach military strategy, is Smedberg (1994). There, explicit fundamental principles of strategies are directly suggested based on different historical examples. The reader who rapidly wants to get the general ideas behind the many different military principles that can be found in the literature, has the option to get exactly that. A problem is however the complete lack of mathematically defined theories and results. To prove that some military principle, or "rule of thumb", is indeed optimal, it is necessary to specify an objective function and some relevant constraints. With mathematics, it would probably be possible to derive optimal military decisions and to understand how these are dependent on possible changes of different kinds of relevant conditions. The lack of mathematics is however common in military instructions in many countries, and one of the main reasons why this manuscript is written.

The many military principles and rules of thumb presented in Smedberg (1994) are at least motivated by many historical examples. Furthermore, the book also gives indirect lectures on military history and geography. Other books on strategy and/or tactics often contain detailed instructions, completely without historical examples and mathematical foundations.

Eriksson and Palmquist (2016) and Palmquist (2016), contain detailed instructions on how to run an army battalion and a brigade. Eriksson (2023) is a recently updated book with instructions on tactical decisions in the army.

1.2. Military analysis with formal methods

The mathematical approaches to military problems and decisions can be split into continuous time modeling, via differential equations, sometimes optimal control and game theory, and discrete time methods. Lohmander (2017) is an introduction to different kinds of mathematical approaches and typical applications. A deep and complete presentation of differential equations and applications is available in Braun (1993). Isaacs (1965) is a great pioneering book on differential games with military applications.

1.2.1. Military analysis with formal methods in continuous time

A key approach within this section is to model the battles via systems of differential equations. Such research may often be descriptive or normative.

1.2.1.1. Descriptive military analysis with formal methods in continuous time

This section focuses on descriptions of the dynamics of historical battles via differential equation systems. The focus of the analysis is to statistically estimate the parameters of the differential equations.

The first well known article in this area is Lanchester (1916). He studied how to model air battles with fighter planes via systems of differential equations. He could analytically show that there are options to improve the battle outcomes via spatial concentration of forces. Later, similar models of military battles were named "Lanchester models". This approach was used to study the dynamics of the Battle of Iwo Jima, during World War II, by Engel (1954). Bracken (1995) used the same approach to model the Ardennes Campaign and so did Tam (1998) and Hung et al (2005). Spradlin and Spradlin (2007) expanded the Lanchester analysis to three dimensions, and Stymfal (2022) updated the analysis of the Battle of Iwo Jima, started by Engel (1954). A new way to estimate the coefficients of the differential equations in Lanchester differential equation systems, via nonlinear iteration, was introduced by Lohmander (2024). As an illustration, he also updated the Iwo Jima model and analysis.

1.2.1.2. Normative military analysis with formal methods in continuous time

Research in this class has the ambition to determine optimal strategic and/or tactical decisions, partly based on differential equation systems that describe the dynamics of battle outcomes, as a functions of the parameters and decisions.

Sheeba and Goose (2008) optimize military resource allocation in battles based on Lanchester models. Chen and Qui (2014) focus on optimal reinforcement in dynamic games and Chan (2016) writes about optimal military force preparations and structure in Singapore. Kress et al (2018) analyze optimal combat with three parties, and Hy et al (2020) optimize Lanchester systems with asymmetries. Kalloniatis et al (2020) optimize the structure in Lanchester models with fire and moves, and McCartney (2022) studies inter- battle reinforcements between repeated Lanchester type battles. Kostic and Jovanovic (2023) use Lanchester models as command decision tools.

The war of Russia on Ukraine has also found Lanchester applications. Lystopadova and Khalaim (2023) apply the Lanchester system solutions to derive optimal military strategies. Lohmander (2023) also applies differential equations to derive optimal strategies in Ukraine during the war caused by the Russian invasion. He uses the differential equations in combination with optimal control theory to determine the optimal time path of the location of the battle front. Lohmander (2025) determines optimal deployment decisions based on a Lanchester battle model in combination with an objective function that considers the costs of deployment, the cost of the time it takes to win the battle, the value of the territory, and the costs of killed and wounded soldiers.

1.2.2. Military analysis with formal methods in discrete time

Here, we find research that is essentially focusing on mathematical analysis and optimization approaches that do not include differential equations.

Typical and central problems in this class of research are found in Washburn (2003). He analyzes many types of decision problems that occur in battles with two parties, fighting each other. The key approach is two- person zero sum game theory. With this method, detailed numerical solutions can mostly easily be found via linear programming, which is useful in several conflicts. Classical applications include sub marine warfare. Another area of research concerns the topic “search and detection”, where Washburn (2002) is a great example. This topic is also of direct relevance to the analysis in this paper, since cavalry rangers may be used to search large areas and find

targets, suitable for attack and destruction by the cavalry rangers, often supported by fire from the artillery. A very well written general description of many kinds of military problems with relevant solutions based on methods from mathematics, statistics and operations research, is found in Washburn and Kress (2009). The classical game theory approach, essentially applied to military problems from air force and naval operations, found din Washburn (2003), serves as a starting point in Lohmander (2019a), where some battle game problems, solutions and proofs are generalized. Furthermore, comparative statics analysis is applied to determine the expected effects of possible errors in parameter estimates. Lohmander (2019b) investigates and solves four concrete and typical military problems in the army, via these methods and solutions defined and described in Lohmander (2019a).

The literature on defense and other military topics contains many detailed instructions on strategies, operations, tactical methods and principles. An important problem is, however, that a general theory that shows how the different strategies, operations and tactical decisions optimally should be adapted to changes in parameters, cannot easily be found. For these reasons, this analysis will contain two mathematical models for explicit optimization of key decisions.

The first model covers the preparations of the defense, and determines the optimal depth of the defense area, the optimal levels of field work for protection and field work for attack, and the optimal initial size of the army deployed to the area. The objective function includes the cost per area unit of defense, the cost per unit of the deployed troops, the costs of the different kinds of field work, the values of the outcomes of the battles, including valuation of the regained land after the war, the numbers of surviving defending soldiers, the numbers of enemy soldiers, and the total time of the war. The model is used to determine how the optimal depth of the defense area, the optimal number of deployed soldiers and the optimal investments in field work for protection, and for attack, are affected by changing prices per land area unit and cost per deployed soldier.

The second model covers the optimal use of different kinds of battalions, such as artillery, cavalry rangers, and mechanized battalions, during the war. Each kind of battalion has constrained resources, and the total outcome of the war is a function of how the different kinds of battalions spend these resources. For instance, cavalry rangers can focus on finding and reporting different kinds of enemy unit locations to the artillery battalion, or to cooperate with mechanized battalions for coordinated ambush. The artillery can focus on fighting enemy artillery, assisting mechanized units in joint attacks on enemy convoys etc.

The attrition levels, for both sides in the conflict, are defined for each kind of battalion, as nonlinear functions of the amounts of time and resources that different kinds of battalions spend on fighting different kinds of enemy battalions. The total system is defined as a controlled nonlinear differential equation system. A discrete time approximation of this differential equation system is used as a subroutine within a numerical global optimization

model. The user can specify the parameters of the objective function, representing the total value of the resources, including the different kinds of battalions, at a selected future point in time. The model may be used to show how the different kinds of battalions should optimally use the available resources. An illustration shows how the Blue Cavalry rangers should optimally use the time and resources, for different values of the parameter representing the expected future cost per Red Artillery battalion, 100 days from the start of the battle.

Figure 1 defines symbols of some of the battalions used in the analyses, and Figure 2 shows typical spatial arrangements of a Brigade defense area, along the lines suggested by Palmquist (2016).

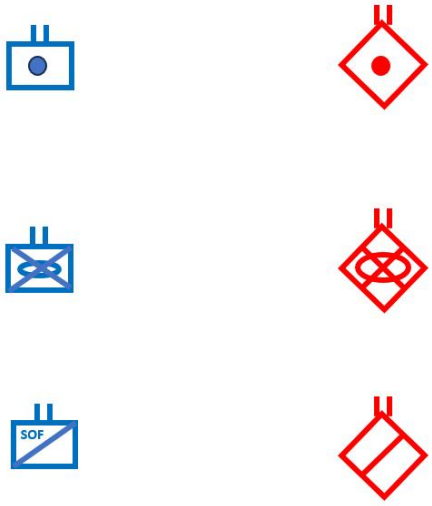


Figure 1.

Symbols of battalions involved in the analysis. The left column, Blue, represents defending battalions, and the right column, Red, are attacking (enemy) battalions. The upper row is artillery, the middle row contains mechanized battalions, and the bottom row shows cavalry (rangers). SOF means special operations forces.

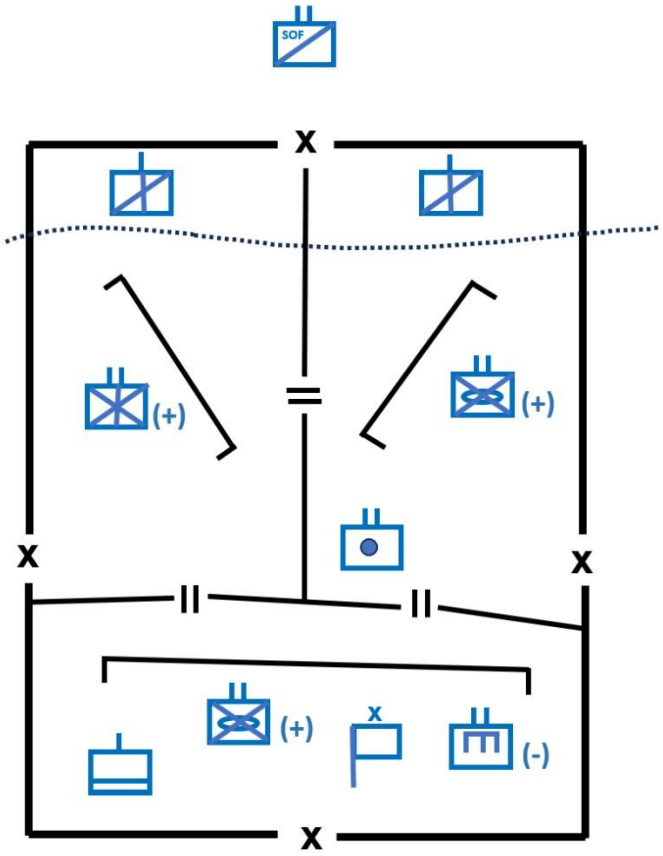


Figure 2.

A brigade defense area. A brigade includes many battalions. Some of these are described in Figure 1. The enemy is expected to attack from the top of the graph, where also the Blue SOF Cavalry Ranger battalion is found. The depth of the defense is the distance from the upper horizontal line (brigade border) with brigade sign "X" to the lower brigade border. In the lower part of the map, the "back area", we find a transport company, a strengthened mechanized battalion, the flag of the brigade commander, and a reduced engineering battalion. Just below the upper brigade border, there are two motorized cavalry ranger companies. Below the left motorized cavalry ranger company, there is also a strengthened motorized infantry battalion.

2. Model 1: Optimal strategic design of the brigade defense

The extended Lanchester simultaneous differential equation system is found in (1).

(1a) describes how the size of the Blue defending force, x, changes over time as a function of the enforcement level, u, and the attrition. The level of attrition of x, is given by the attrition coefficient α multiplied by the size of the Red enemy force, y.

(1b) describes how the size of the Red attacking (enemy) force, y, changes over time as a function of the enforcement level, v, and the attrition. The level of attrition of y, is given by the attrition coefficient β multiplied by the size of the Blue defending force, x.

$$\begin{cases} \frac{dx}{dt} = u - \alpha y & (1a) \\ \frac{dy}{dt} = v - \beta x & (1b) \end{cases} \tag{1}$$

Initial assumption of constant reinforcement levels: Let $\frac{du}{dt} = \frac{dv}{dt} = 0$. Equation (1a) gives (2). The time derivative of (2) is found in (3).

$$y = \frac{1}{\alpha}\left(u - \frac{dx}{dt}\right) \tag{2}$$

$$\frac{dy}{dt} = -\alpha^{-1}\frac{d^2x}{dt^2} \tag{3}$$

Equations (1b) and (3) lead to (4). This is transformed to the second order differential function (5), which is modified to (6).

$$v - \beta x = -\alpha^{-1}\frac{d^2x}{dt^2} \tag{4}$$

$$\frac{d^2x}{dt^2} - \alpha\beta x + \alpha v = 0 \tag{5}$$

$$\frac{d^2x}{dt^2} - \alpha\beta x = -\alpha v \tag{6}$$

Homogenous solution:

$$\frac{d^2 x_H}{dt^2} - \alpha\beta x_H = 0 \tag{7}$$

Let $x_H = Ae^{\lambda t}$

$$\lambda^2 Ae^{\lambda t} - \alpha\beta Ae^{\lambda t} = 0 \tag{8}$$

$$\begin{gathered}\left(\lambda^2 - \alpha\beta\right) Ae^{\lambda t} = 0 \\ \left(=0\right) \quad \left(\neq 0\right)\end{gathered} \tag{9}$$

The attrition coefficients are both strictly positive, which leads to (10). As a result, there are two distinctly different real roots to the equation, as found in (11), which implies that the homogenous solution has the functional form found in (12).

$$\lambda^2 = \alpha\beta > 0 \tag{10}$$

$$\lambda = \pm\sqrt{\alpha\beta} \tag{11}$$

$$x_H = A_1 e^{\lambda t} + A_2 e^{-\lambda t} \quad , \lambda = \sqrt{\alpha\beta} \tag{12}$$

Particular solution:

$$\frac{d^2 x_P}{dt^2} - \alpha\beta x_P = -\alpha v \tag{13}$$

Let $x_P = m$

$$-\alpha\beta m = -\alpha v \tag{14}$$

As a result, we get the particular solution (15).

$$x_P = \frac{v}{\beta} \tag{15}$$

Solution:

The solution (16) is the sum of the homogenous and the particular solutions. This is further specified in (17).

$$x(t) = x_H(t) + x_P(t) \tag{16}$$

$$x(t) = A_1 e^{\lambda t} + A_2 e^{-\lambda t} + \frac{v}{\beta} \tag{17}$$

Equation (1a) gives (18), which contains the time derivative of (17), found in (19).

$$y = \frac{1}{\alpha}\left(u - \frac{dx}{dt}\right) \tag{18}$$

$$\frac{dx}{dt} = \lambda A_1 e^{\lambda t} - \lambda A_2 e^{-\lambda t} \tag{19}$$

Equations (18) and (19) lead to (20), which is further developed to (21).

$$y(t) = \frac{1}{\alpha}\left(u - \left(\lambda A_1 e^{\lambda t} - \lambda A_2 e^{-\lambda t}\right)\right) \tag{20}$$

$$y(t) = -\frac{\lambda}{\alpha} A_1 e^{\lambda t} + \frac{\lambda}{\alpha} A_2 e^{-\lambda t} + \frac{u}{\alpha} \tag{21}$$

The two equations (22) and (23), are the solutions to $\left(x(t), y(t)\right)$.

$$x(t) = A_1 e^{\lambda t} + A_2 e^{-\lambda t} + \frac{v}{\beta} \tag{22}$$

$$y(t) = -\frac{\lambda}{\alpha} A_1 e^{\lambda t} + \frac{\lambda}{\alpha} A_2 e^{-\lambda t} + \frac{u}{\alpha} \tag{23}$$

Equation (11) gives more explicit version of (23) in equations (24) and (25).

$$y(t) = -\frac{\sqrt{\alpha\beta}}{\alpha} A_1 e^{\lambda t} + \frac{\sqrt{\alpha\beta}}{\alpha} A_2 e^{-\lambda t} + \frac{u}{\alpha} \tag{24}$$

$$y(t) = -\frac{\sqrt{\beta}}{\sqrt{\alpha}} A_1 e^{\lambda t} + \frac{\sqrt{\beta}}{\sqrt{\alpha}} A_2 e^{-\lambda t} + \frac{u}{\alpha} \tag{25}$$

Now, as we see in (26) and (27), thanks to Cramer´s rule, the values of the parameters $\left(A_1, A_2\right)$ can be determined from the two simultaneous equation systems.

$$\begin{bmatrix} e^{\lambda t} & e^{-\lambda t} \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{\lambda t} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{-\lambda t} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} x(t) - \frac{v}{\beta} \\ y(t) - \frac{u}{\alpha} \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} e^{\lambda t} & e^{-\lambda t} \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{\lambda t} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{-\lambda t} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} x(t) - \frac{v}{\beta} \\ y(t) - \frac{u}{\alpha} \end{bmatrix} \tag{27}$$

The parameters $\left(A_1, A_2\right)$ are found as explicit functions of the parameters in (28) and (29). The solutions, based on the initial conditions, are found in (30) and (31).

$$A_1 = \frac{\begin{vmatrix} x(t) - \frac{v}{\beta} & e^{-\lambda t} \\ y(t) - \frac{u}{\alpha} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{-\lambda t} \end{vmatrix}}{\begin{vmatrix} e^{\lambda t} & e^{-\lambda t} \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{\lambda t} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{-\lambda t} \end{vmatrix}} \tag{28}$$

$$A_2 = \frac{\begin{vmatrix} e^{\lambda t} & x(t) - \frac{v}{\beta} \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{\lambda t} & y(t) - \frac{u}{\alpha} \end{vmatrix}}{\begin{vmatrix} e^{\lambda t} & e^{-\lambda t} \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{\lambda t} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{-\lambda t} \end{vmatrix}} \tag{29}$$

Determination of the parameters A_1 and A_1 via the initial conditions (t = 0):

$$A_1 = \frac{\begin{vmatrix} x(0) - \frac{v}{\beta} & 1 \\ y(0) - \frac{u}{\alpha} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} \end{vmatrix}} = \frac{\left(x(0) - \frac{v}{\beta} \right) \frac{\sqrt{\beta}}{\sqrt{\alpha}} - y(0) + \frac{u}{\alpha}}{2\frac{\sqrt{\beta}}{\sqrt{\alpha}}} \tag{30}$$

$$A_2 = \frac{\begin{vmatrix} 1 & x(0) - \frac{v}{\beta} \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} & y(0) - \frac{u}{\alpha} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} \end{vmatrix}} = \frac{y(0) - \frac{u}{\alpha} + \frac{\sqrt{\beta}}{\sqrt{\alpha}} \left(x(0) - \frac{v}{\beta} \right)}{2\frac{\sqrt{\beta}}{\sqrt{\alpha}}} \tag{31}$$

The analytically determined equations for the sizes of the two armies are included in the optimization software found in Appendix 1. There, the attrition parameters are determined as functions of the depth of the defense, the level of field work for attack, FWA, and field work for protection, FWP.

Optimization with Model 1: Optimal spatial design of the brigade defense

The optimization software is found in Appendix 1. There, all parameters and other details may be found.

The objective function of Blue is maximized. This is found in equation (32). Note that the parameters values in (32) may easily be adjusted to other relevant values, based on locally relevant conditions, in specific applications.

$$\begin{aligned}\Pi = & -P_{Depth} \times Depth - P_{x_0} \times x_0 - .0002 \times (x_0)^2 - 1 \times u \times T \\ & -170 \times FWP - 170 \times FWA + e^{-rT}(x_T - y_T) \\ & -10000 \times ywins\end{aligned} \tag{32}$$

The objective function can be explained this way: It is a decreasing function of $Depth$, the depth of the defense and a decreasing and strictly concave function of x_0, the initial number of defending soldiers. It is also a decreasing function of the rate of reinforcement, u, and of the total time of the war, T. The objective function is also a decreasing function of FWP, the level of field work for protection, and of FWA, the level of field work for attack. The difference $(x_T - y_T)$ shows how much larger the BLUE force is than the RED force, when the war ends. The objective function is an increasing function of this difference. In the objective function, this difference is discounted, as a decreasing function of time. The "war rate of interest", r, describes how rapidly the values of the resources, per unit, are reduced during the war. BLUE or RED will win the war. If RED wins, then $ywins$ equals one. $ywins$ equals zero if y_T equals zero. Clearly, the BLUE objective function is a decreasing function of $ywins$.

The optimal depth of the defense and the optimal initial number of defending soldiers, as decreasing functions of the respective prices.

The numerically obtained optimal results are found in Figure 3. and in Figure 4.

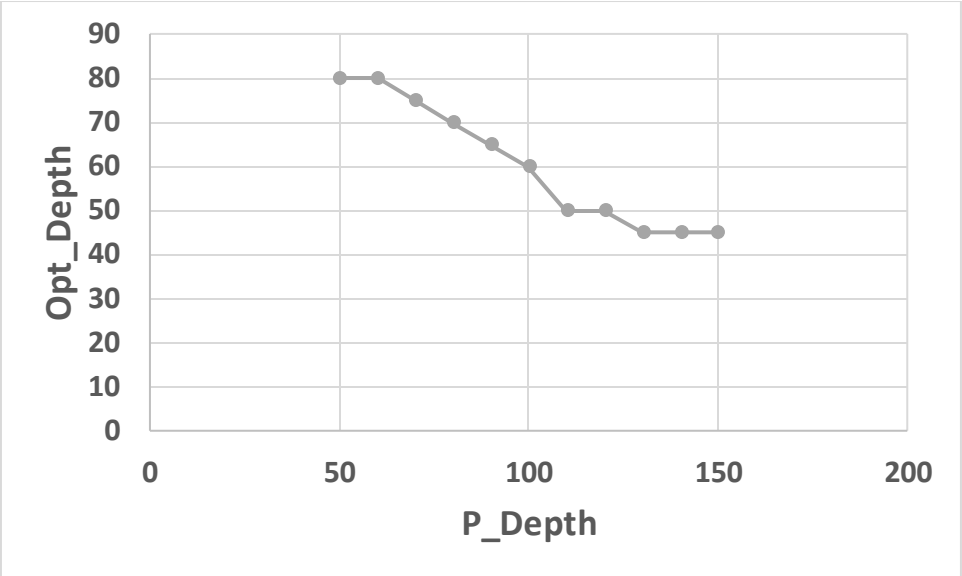


Figure 3.
The optimal depth of the defense is a decreasing function of the price of the land.

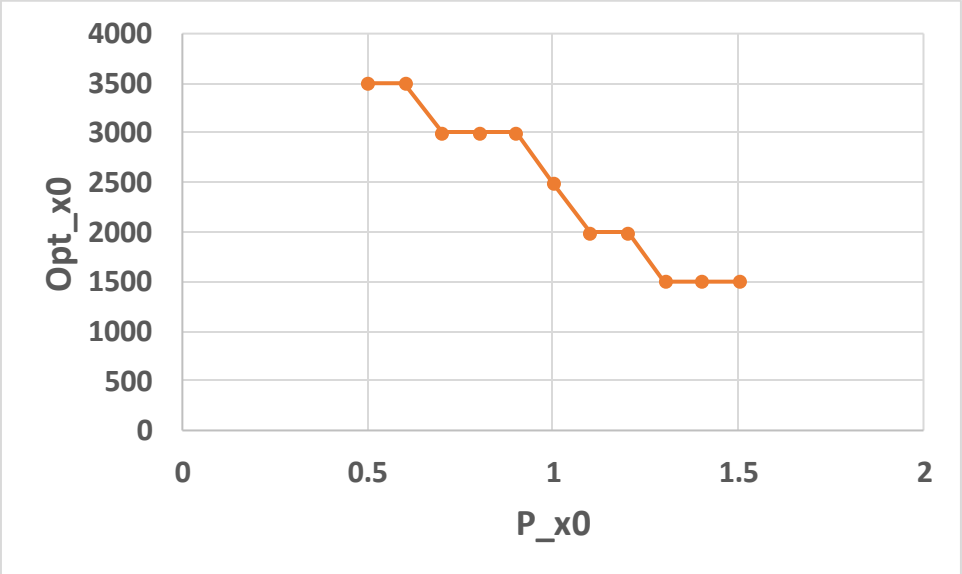


Figure 4.
The optimal number of soldiers in the initial force is a decreasing function of the price per soldier.

The optimal level of field work for protection, FWP is an increasing function of the price of land and of the price of soldiers. This is shown in Figures 5 and Figure 6.

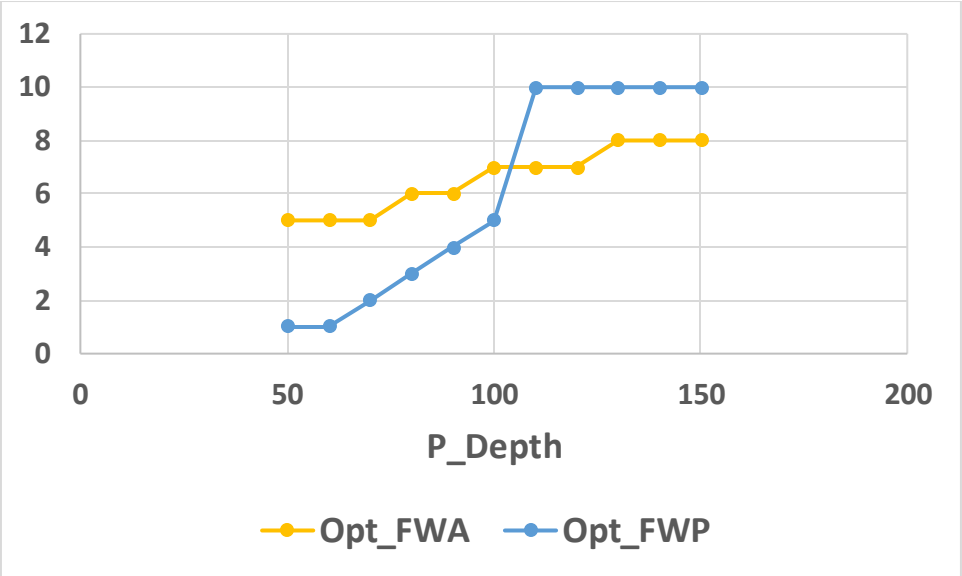


Figure 5.

If the price of the land increases, the optimal depth of the defense is reduced, as we see in Figure 4. Then, it is optimal to partly increase the field work for attack, FWA, and strongly increase the field work for protection, FWP.

Figure 6 shows that the optimal level of field work for attack, FWA, is an increasing function of the price of land.

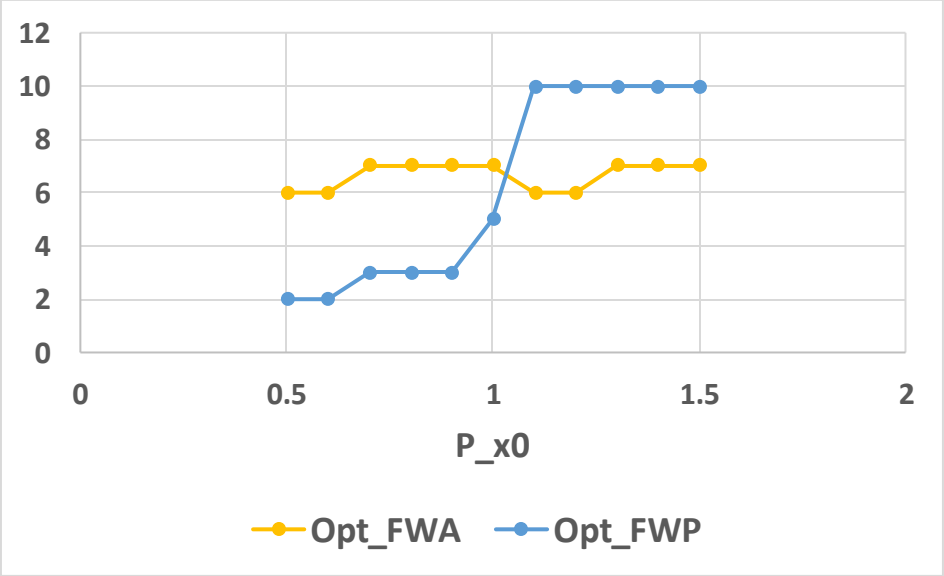


Figure 6.
The optimal level of the field work for protection is an increasing function of the price per soldier in the initial force. The optimal level of the field work for attack is not sensitive to the price per soldier.

The point in time when the war ends, Opt_T, is strongly dependent on the parameters and the conditional optimal decisions. This is clear from Figure 7 and Figure 8.

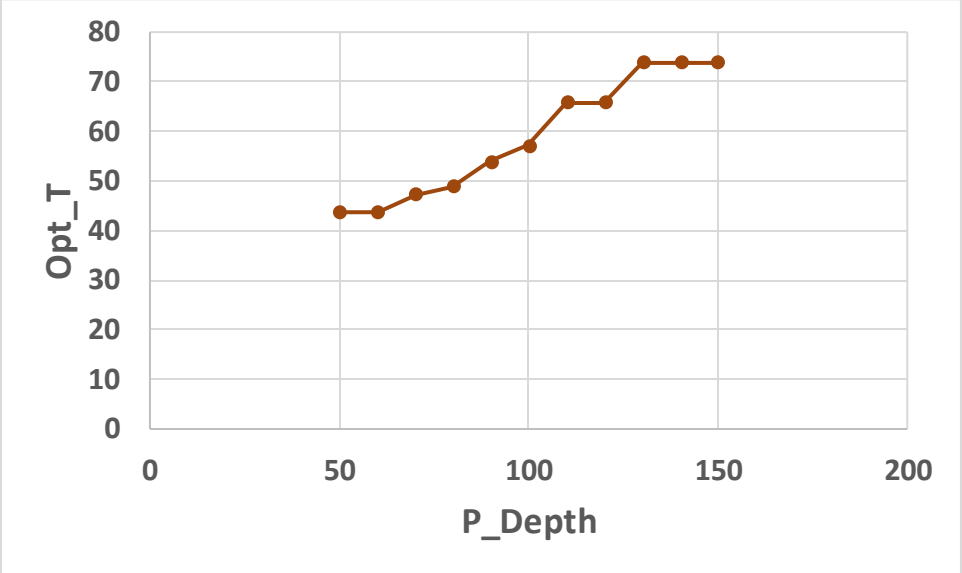


Figure 7.
If the price of the land increases, the war goes on for a longer time.

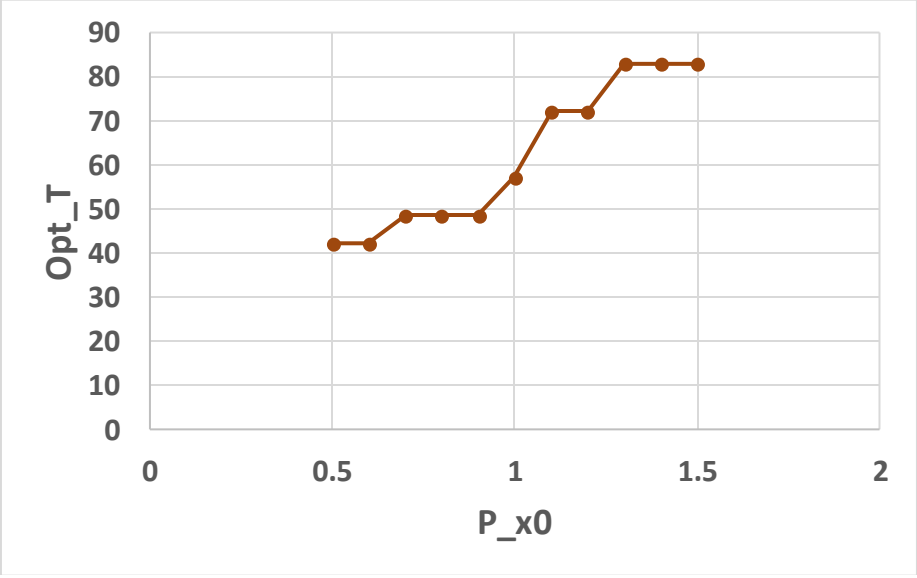


Figure 8.
The time when the war stops is a strongly increasing function of the price per soldier.

The optimal number of surviving and unharmed soldiers after the end of the war, x_T, is a function of the parameters and the optimal decisions. Figure 9 and Figure 10 demonstrate these results.

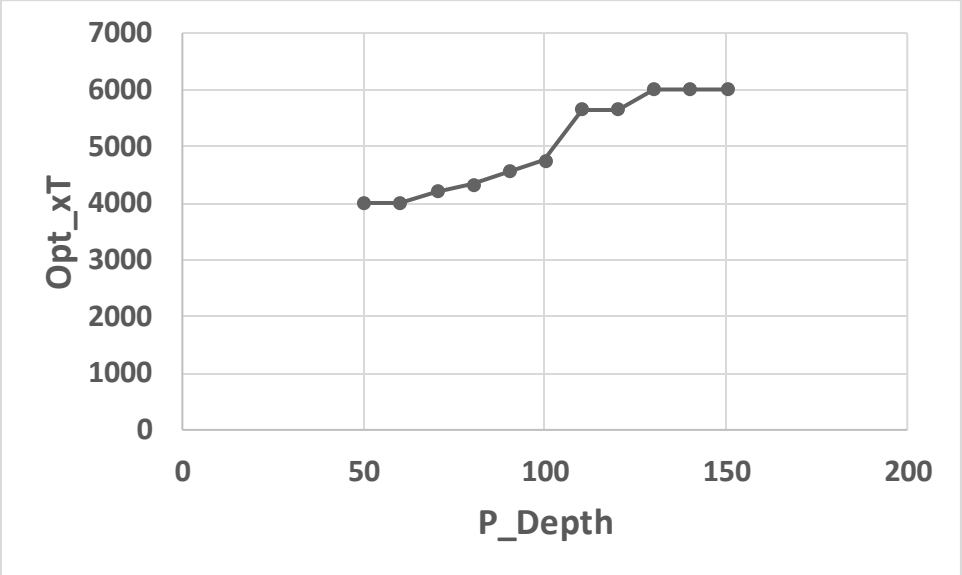


Figure 9.

If the price of the land increases, the number of soldiers that eventually return from the war increases. Two reasons are that the field work for protection increases (Figure 5), which increases the probability to survive (Figure 6), and more soldiers are sent to the front before the war ends, since the war goes on for a longer time (Figure 8).

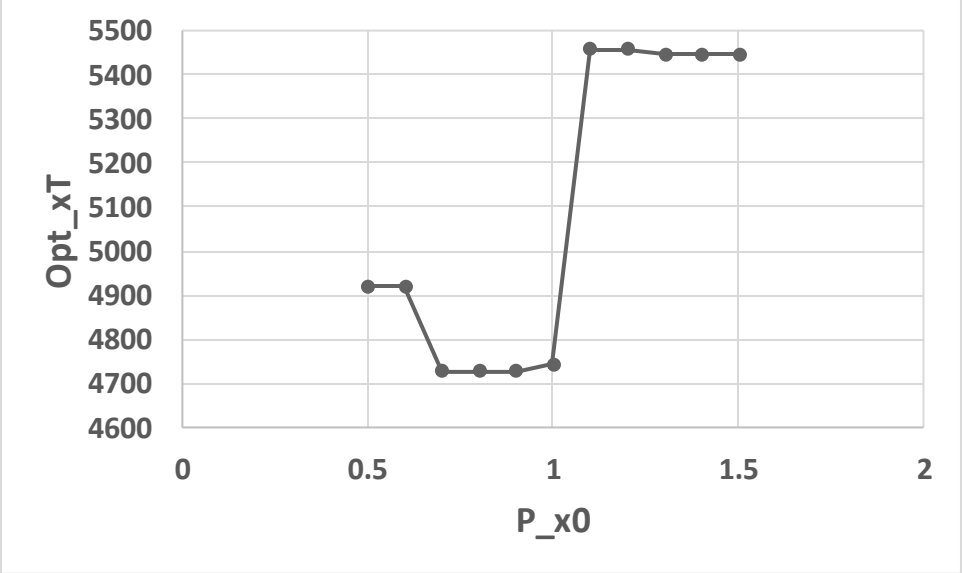


Figure 10.

The size of the force that returns home from the war is a U shaped function of the price per soldier.

Figure 11 and Figure 12 illustrate that the optimal objective function value, Opt_Obj, is a decreasing function of the price of land and the price per defending soldier.

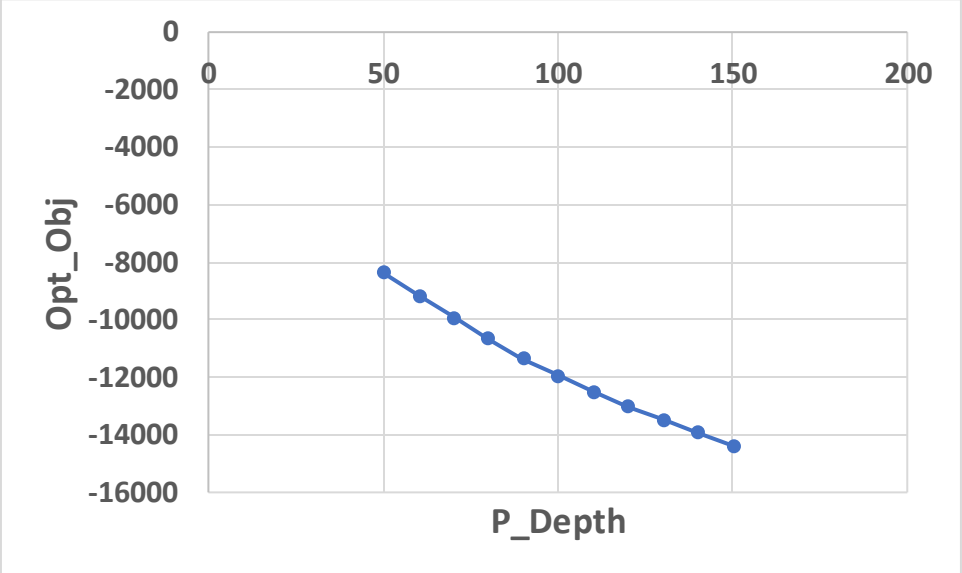


Figure 11.
The value of the optimized objective function is a decreasing function of the price of the land.

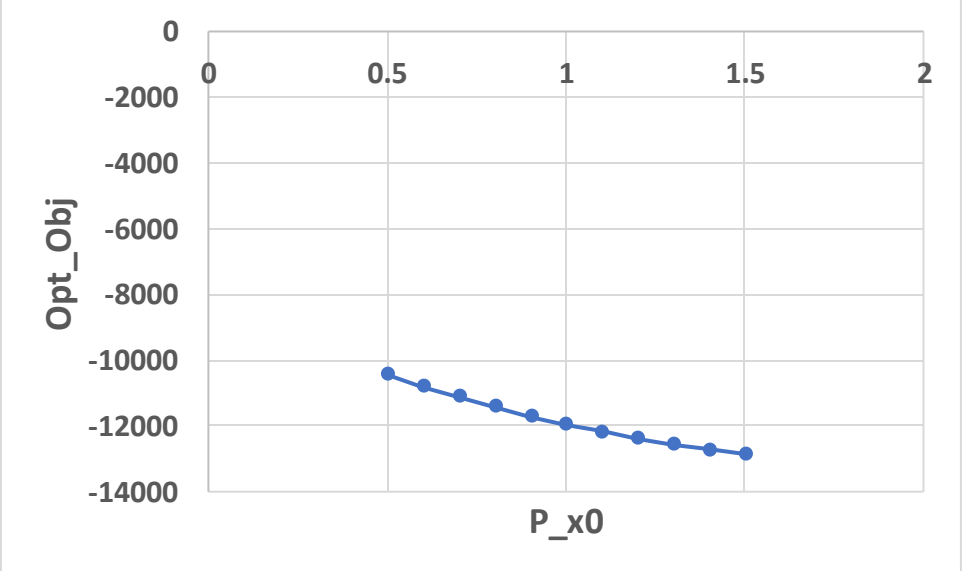


Figure 12.

The value of the optimized objective function is a decreasing function of the price per soldier in the initial force.

In Figure 13 and Figure 14, we find that the optimal value of α, the attrition coefficient that determines how rapidly the defending force is depleted, is reduced in case the price of land increases and/or if the price per defending soldier increases. Figure 13 also shows that the optimal value of β, the attrition coefficient that determines how rapidly the attacking force is depleted, is reduced in case the price of land increases.

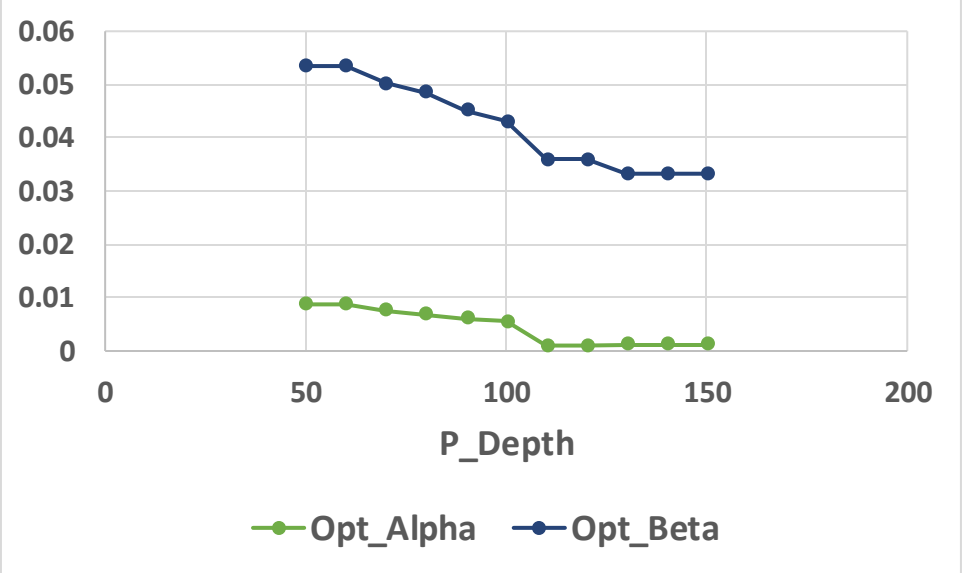


Figure 13.
If the price of the land increases, the depth of the defense is reduced. The attrition parameter Alpha is very much reduced, since FWP is strongly increased. The parameter Beta is also reduced, but not as strongly as Alpha.

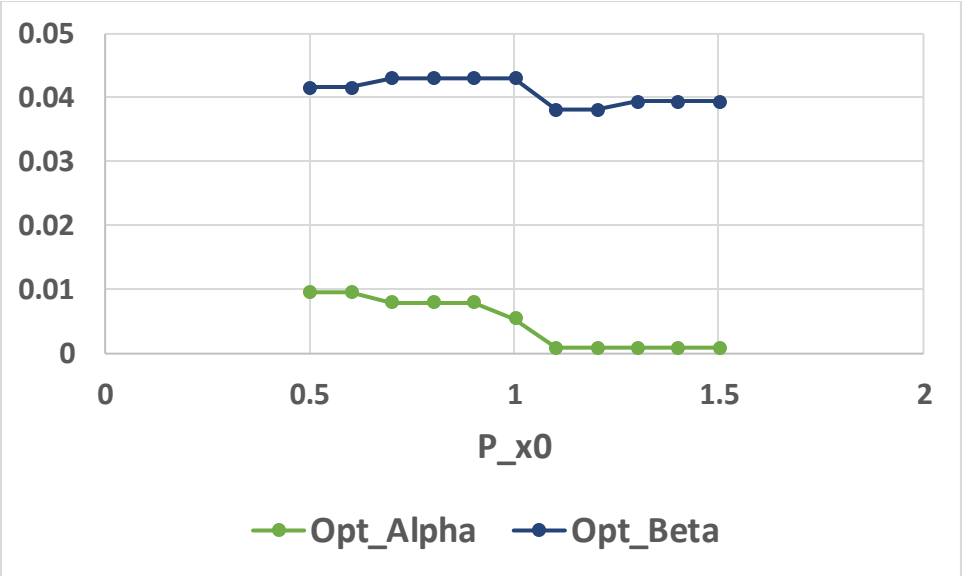


Figure 14.
The optimal level of Alfa is very sensitive to the price per soldier. If the price per soldier increases, the optimal number of soldiers strongly decreases. However, since the optimal field work for protection strongly increases, the optimal value of Alpha strongly decreases, and the probability that the soldiers survive strongly increases. The optimal level of Beta is not sensitive to the price per soldier.

3. Model 2: Optimal coordination within a brigade

This section focuses on battles with brigades, with more details. Several principles of battles with army brigades, are described in Palmquist (2016). One key idea is that a defending brigade, BLUE, should try to separate different units, such as battalions, in the attacking brigade, RED. Figure 15 shows such a situation. One BLUE battalion stops two RED battalions, when they are moving on a road. The first red battalion can then be isolated and attacked by two BLUE battalions. From the analytical theory based on Lanchester models, it is well known that it is important to concentrate forces. Compare Lanchester (1916) and Braun (1993). In Figure 15, all battalions are mechanized. In Figure 16, the situation is more complicated.

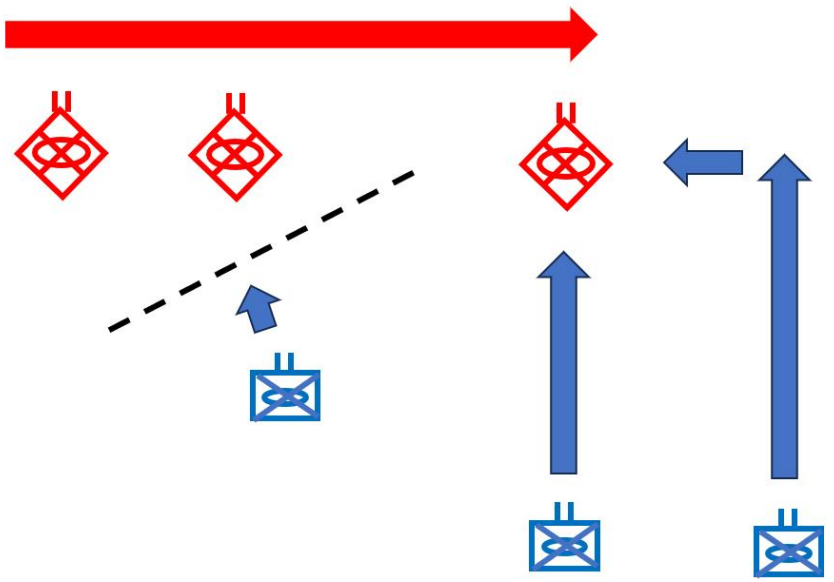


Figure 15.
Blue assault on RED invaders moving on a road from left to right. One BLUE mechanized battalion stops the two last RED mechanized battalions. Simultaneously, two BLUE battalions jointly attack the first RED battalion.

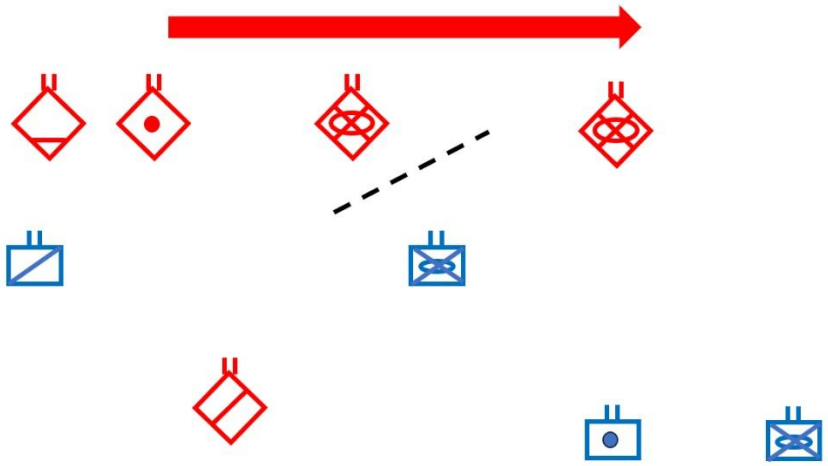


Figure 16.
Alternative situation before a BLUE assault on RED invaders starts. The RED force contains two mechanized battalions, one artillery battalion and one transport battalion. These move in this order from left to right. In the terrain, down, to the left, a red cavalry ranger battalion is found. The BLUE forces contain two mechanized battalions, one artillery battalion and one cavalry ranger battalion.

Figure 17 shows how the BLUE artillery battalion can fight three different kinds of enemy targets; A mechanized battalion, an artillery battalion and/or a cavalry ranger battalion.

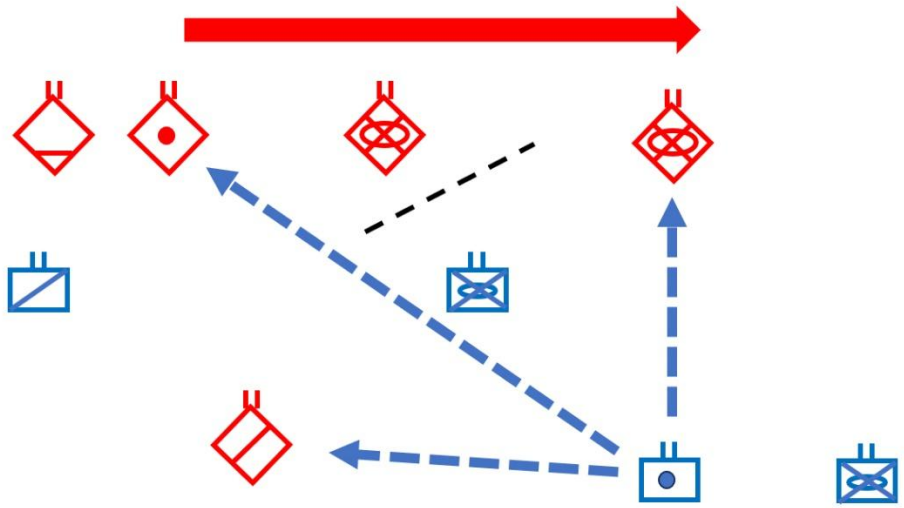


Figure 17.
The Blue assault on RED invaders starts. The BLUE artillery battalion attacks the first red mechanized battalion, the red artillery battalion and the red cavalry ranger battalion.

Figure 18 demonstrates how the BLUE cavalry ranger battalion can attack the RED artillery battalion, the transport battalion and/or the cavalry ranger battalion.

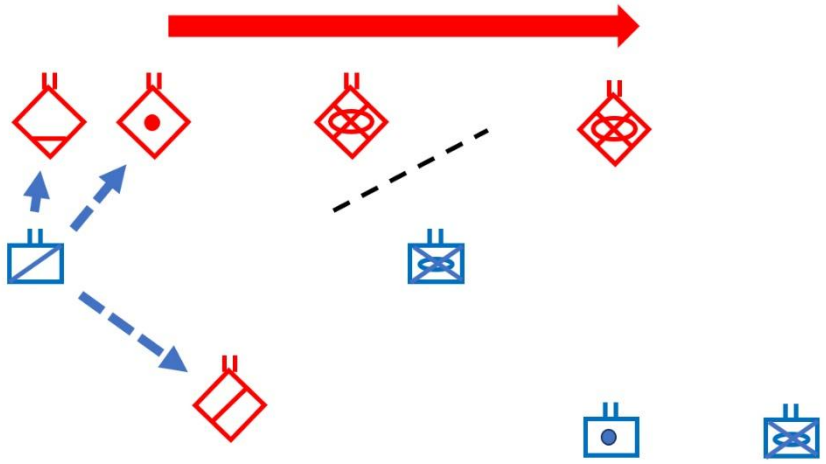


Figure 18.
The BLUE assault on RED forces continues, with continued indirect fire support from the BLUE artillery battalion. The BLUE cavalry ranger battalion attacks the RED artillery battalion and the transport battalion on the move, and the RED cavalry ranger battalion.

In Figure 19, one BLUE mechanized battalion stops one of the RED mechanized battalions. At the same time, another BLUE mechanized battalion strikes the first RED mechanized battalion.

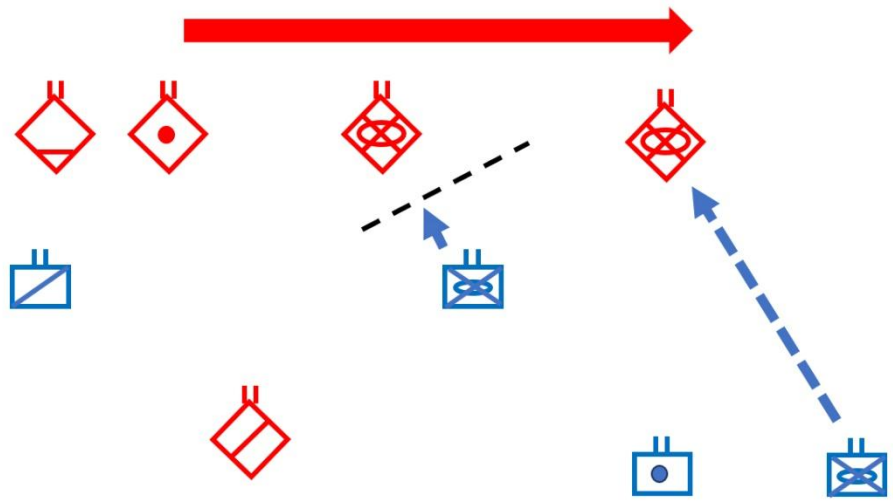


Figure 19.
The BLUE assault on RED forces continues, with continued indirect fire support from the BLUE artillery battalion. The BLUE mechanized battalions stop one RED mechanized battalion and the other BLUE mechanized battalion attacks the first RED mechanized battalion.

Figures 15 to 19 have shown how different kinds of battalions may attack different kinds of RED battalions. In Figures 20 and 21, the possible attacks have been summarized via the arrows. We may consider these arrows to represent the shares of total time that the different kinds of battalions spend on each kind of attack. In the rest of the analysis, we will focus on the determination of the optimal levels of the BLUE shares of time. In this analysis, we consider the use of time by the battalions in the RED army to be exogenously determined. It is however possible to expand the analysis of the system to a complete game theory setting. Then, the RED army should also optimize the use of time for each kind of battalion. That kind of analysis may perhaps appear in a later article.

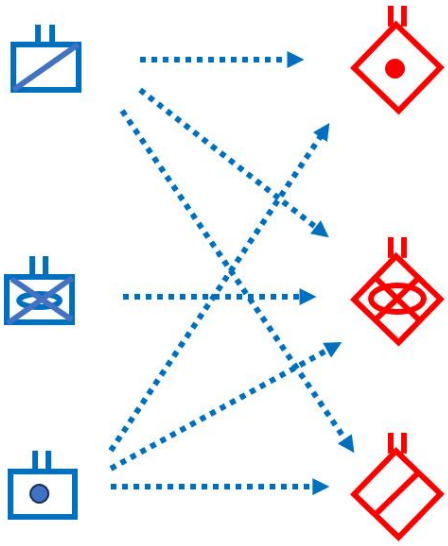


Figure 20.
The possible attacks by BLUE battalions on RED battalions are represented by the blue arrows. These arrows represent the shares of total time that the different kinds of battalions spend on each kind of attack.

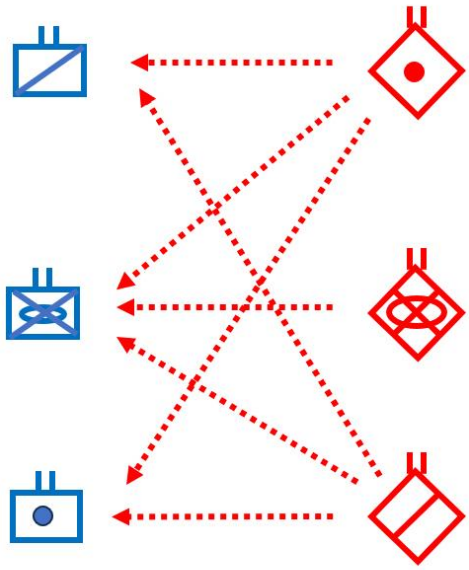


Figure 21.
The possible attacks by RED battalions on BLUE battalions are represented by the red arrows. These arrows represent the shares of total time that the different kinds of battalions spend on each kind of attack.

Dynamic model

In this section, a model with interactions between three kinds of battalions in the BLUE and the RED army will be studied. The decisions in the BLUE army will be optimized. The sizes of the artillery, the cavalry rangers and mechanized forces, in the units "full battalions", in the BLUE army, are denoted (X_A, X_C, X_M). These sizes are functions of time. The corresponding sizes of the units in the RED army are (Y_A, Y_C, Y_M).

The shares of time that the BLUE artillery spends on fighting RED artillery, cavalry rangers and mechanized units, are (u_{AA}, u_{AC}, u_{AM}). The shares of time that the RED artillery spends on fighting BLUE artillery, cavalry rangers and mechanized units, are (v_{AA}, v_{AC}, v_{AM}). In the same way, the other kinds of units spend time on fighting different kinds of enemy units.

The attrition of a particular kind of unit is a nonlinear function of the Cobb Douglas type, of the amounts of time that different kinds of battalions spend in this activity. For instance, if the size of the BLUE artillery is not constraining the attrition of the BLUE artillery, then $G_A\left(v_{AA}Y_A(t)\right)^{r_{AA}}\left(v_{CA}Y_C(t)\right)^{r_{CA}}$ is the attrition of the BLUE artillery, as a function of the amount of RED artillery and RED cavalry rangers that are used in this kind of operation. (r_{AA}, r_{CA}) are two exponents. With constant returns to scale, the sum of these exponents is one.

In the same way, if the size of the RED artillery is not constraining the attrition of the RED artillery, then $H_A\left(u_{AA}X_A(t)\right)^{s_{AA}}\left(u_{CA}X_C(t)\right)^{s_{CA}}$ is the attrition of the RED artillery, as a function of the amount of BLUE artillery and BLUE cavalry rangers that are used in this kind of operation.

Dynamically changing constrained attrition levels based on particular production functions in continuous time

Equations (33) to (38) give the attrition levels of the BLUE and RED battalions of different kinds. Note that these are given in continuous time.

$$\psi_{X_A}(t) = \min\left\{X_A(t), G_A\left(v_{AA}Y_A(t)\right)^{r_{AA}}\left(v_{CA}Y_C(t)\right)^{r_{CA}}\right\} \tag{33}$$

$$\psi_{X_C}(t) = \min\left\{X_C(t), G_C\left(v_{AC}Y_A(t)\right)^{r_{AC}}\left(v_{CC}Y_C(t)\right)^{r_{CC}}\right\} \tag{34}$$

$$\psi_{X_M}(t) = \min\left\{X_M(t), G_M\left(v_{AM}Y_A(t)\right)^{r_{AM}}\left(v_{CM}Y_C(t)\right)^{r_{CM}}\left(v_{MM}Y_M(t)\right)^{r_{MM}}\right\} \tag{35}$$

$$\psi_{Y_A}(t) = \min\left\{Y_A(t), H_A\left(u_{AA}X_A(t)\right)^{s_{AA}}\left(u_{CA}X_C(t)\right)^{s_{CA}}\right\} \tag{36}$$

$$\psi_{Y_C}(t) = \min\left\{Y_C(t), H_C\left(u_{AC}X_A(t)\right)^{s_{AC}}\left(u_{CC}X_C(t)\right)^{s_{CC}}\right\} \tag{37}$$

$$\psi_{Y_M}(t) = \min\left\{Y_M(t), H_M\left(u_{AM}X_A(t)\right)^{s_{AM}}\left(u_{CM}X_C(t)\right)^{s_{CM}}\left(u_{MM}X_M(t)\right)^{s_{MM}}\right\} \tag{38}$$

Differential equation system representing the resources in continuous time

The time derivatives, the dynamic changes of the sizes of the different kinds of forces in the two armies, are found in equations (39) to (44). Here, the reinforcement level at time t of the BLUE artillery per time unit, is denoted $iX_A(t)$. The other reinforcement levels are denoted in the corresponding way.

$$\dot{X}_A = iX_A(t) - \psi_{X_A}\left(v_{AA}, v_{CA}, Y_A, Y_C\right) \tag{39}$$

$$\dot{X}_C = iX_C(t) - \psi_{X_C}\left(v_{AC}, v_{CC}, Y_A, Y_C\right) \tag{40}$$

$$\dot{X}_M = iX_M(t) - \psi_{X_M}\left(v_{AM}, v_{CM}, v_{MM}, Y_A, Y_C, Y_M\right) \tag{41}$$

$$\dot{Y}_A = iY_A(t) - \psi_{Y_A}\left(u_{AA}, u_{CA}, X_A, X_C\right) \tag{42}$$

$$\dot{Y}_C = iY_C(t) - \psi_{Y_C}\left(u_{AC}, u_{CC}, X_A, X_C\right) \tag{43}$$

$$\dot{Y}_M = iY_M(t) - \psi_{Y_M}\left(u_{AM}, u_{CM}, u_{MM}, X_A, X_C, X_M\right) \tag{44}$$

Note:

In some equations, the time indices are removed, to simplify notation. In continuous time, the attrition functions have the symbol ψ, and in discrete time, they have the symbol ϕ.

Difference equation system representing the resources in discrete time

Because of the large number of equations and the nonlinearities, it is not possible to obtain explicit analytically solutions. For this reason, numerical approximations are determined via difference equations. $jX_A(t)$ is the discrete time analogue of $iX_A(t)$. The attrition ϕ, in discrete time, corresponds to ψ, in continuous time. Equations (45) to (50) show the differences of the resource system.

$$dX_A(t) = jX_A(t) - \phi_{X_A}(v_{AA}, v_{CA}, Y_A, Y_C) \tag{45}$$

$$dX_C(t) = jX_C(t) - \phi_{X_C}(v_{AC}, v_{CC}, Y_A, Y_C) \tag{46}$$

$$dX_M(t) = jX_M(t) - \phi_{X_M}(v_{AM}, v_{CM}, v_{MM} Y_A, Y_C, Y_M) \tag{47}$$

$$dY_A(t) = jY_A(t) - \phi_{Y_A}(u_{AA}, u_{CA}, X_A, X_C) \tag{48}$$

$$dY_C(t) = jY_C(t) - \phi_{Y_C}(u_{AC}, u_{CC}, X_A, X_C) \tag{49}$$

$$dY_M(t) = jY_M(t) - \phi_{Y_M}(u_{AM}, u_{CM}, u_{MM}, X_A, X_C, X_M) \tag{50}$$

Initial conditions

The long run strategic optimization includes the determination of the resources later defined as the initial conditions, namely $\left(X_A^0, X_C^0, X_M^0\right)$ and $\left(Y_A^0, Y_C^0, Y_M^0\right)$. Compare (51) and (52).

$$\left(X_A(0), X_C(0), X_M(0)\right) = \left(X_A^0, X_C^0, X_M^0\right) \tag{51}$$

$$\left(Y_A(0), Y_C(0), Y_M(0)\right) = \left(Y_A^0, Y_C^0, Y_M^0\right) \tag{52}$$

Recursive determination of the resource dynamics

The difference equations (45) to (50), and the initial conditions, (51) and (52), are used to determine the resources at future points in time (53) to (58).

$$X_A(t+1) = X_A(t) + jX_A(t) - \phi_{X_A}(v_{AA}, v_{CA}, Y_A(t), Y_C(t)) \tag{53}$$

$$X_C(t+1) = X_C(t) + jX_C(t) - \phi_{X_C}(v_{AC}, v_{CC}, Y_A(t), Y_C(t)) \tag{54}$$

$$X_M(t+1) = X_M(t) + jX_M(t) - \phi_{X_M}(v_{AM}, v_{CM}, v_{MM} Y_A(t), Y_C(t), Y_M(t)) \tag{55}$$

$$Y_A(t+1) = Y_A(t) + jY_A(t) - \phi_{Y_A}(u_{AA}, u_{CA}, X_A(t), X_C(t)) \tag{56}$$

$$Y_C(t+1) = Y_C(t) + jY_C(t) - \phi_{Y_C}(u_{AC}, u_{CC}, X_A(t), X_C(t)) \tag{57}$$

$$Y_M(t+1) = Y_M(t) + jY_M(t) - \phi_{Y_M}(u_{AM}, u_{CM}, u_{MM}, X_A(t), X_C(t), X_M(t)) \tag{58}$$

Determination of the tactical control decisions via maximization of the objective functions

In a global optimization model found in the appendix, the BLUE control decisions, the shares of time used by different battalions to fight different kinds of RED battalions, are determined. It is possible to generalize the analysis by also determining the optimal decisions by the RED army. That is however not made in this article.

The optimization problem from the BLUE perspective is given in (59) and the corresponding problem from the RED perspective is found in (60).

$$\begin{aligned} W_1^* = \max_{\substack{u_{AA},u_{AC},u_{AM},\\ u_{CA},u_{CC},u_{CM}}} & W_1(u_{AA}, u_{AC,}u_{AM}, u_{CA}, u_{CC}, u_{CM}; \bullet) \\ & s.t. \\ & u_{AA} + u_{AC} + u_{AM} \le 1 \\ & u_{CA} + u_{CC} + u_{CM} \le 1 \end{aligned} \tag{59}$$

$$\begin{aligned} W_2^* = \max_{\substack{v_{AA},v_{AC},v_{AM},\\ v_{CA},v_{CC},v_{CM}}} & W_2(v_{AA}, v_{AC,}v_{AM}, v_{CA}, v_{CC}, v_{CM}; \bullet) \\ & s.t. \\ & v_{AA} + v_{AC} + v_{AM} \le 1 \\ & v_{CA} + v_{CC} + v_{CM} \le 1 \end{aligned} \tag{60}$$

Examples of objective functions

Equations (61) and (62) shows a simplified example of a possible BLUE objective function. T denotes a selected future point in time, for instance 100 days from the start of the war.

$$W_1 = f_{Y_A} Y_A(T) + f_{Y_C} Y_C(T) + f_{Y_M} Y_M(T) \tag{61}$$

$$(f_{Y_A}, f_{Y_C}, f_{Y_M}) = (-10, -5, -10) \tag{62}$$

Equations (63) and (64) shows a simplified example of a possible RED objective function.

$$W_2 = f_{X_A} X_A(T) + f_{X_C} X_C(T) + f_{X_M} X_M(T) \tag{63}$$

$$\left(f_{X_A}, f_{X_C}, f_{X_M}\right) = (-10, -5, -10) \tag{64}$$

Results from Model 2: Optimal coordination within a brigade

Results derived via the software Opt_Brigade_260227_1444.bas found in Appendix 2, are reported in Tables 1, 2 and 3.

Table 1 shows the optimal use of the cavalry rangers in case it is very important to reduce the number of RED artillery units during the first 100 days. The parameter fYa = -15. The cavalry should spend 75% of the time (u_{ca}) fighting RED artillery, 3% of the time fighting enemy cavalry (u_{cc}) and 22% of the time fighting mechanized units (u_{cm}). After 100 days, BLUE will have approximately 10 units of artillery, 13 units of cavalry and 41 mechanized units. At the same point in time RED will have about 10 units of artillery, 29 units of cavalry and 79 mechanized units. With these results, the optimized BLUE objective function is approximately -1085.

In case the RED artillery units contribute with a less negative value to the BLUE objective function, then the BLUE cavalry should spend less time fighting RED artillery and more time fighting RED cavalry and mechanized units. This is also seen in Table 2 and in Table 3. Furthermore, the BLUE optimal objective function improves. At time T = 100, RED will have more artillery left, but less cavalry and a lower number of mechanized units.

Table 1.

```
Parameter fYa = -15

  Optimal Solution:

   objective function = -1085.318

   uca       ucc       ucm
   0.75      0.03      0.22

             Art       Cav       Mec
X(topt) =  9.557317  12.77866  40.5522
Y(topt) =  9.670883  29.32049  79.3652
```

Table 2.

```
Parameter fYa = -10

  Optimal Solution:

   objective function = -1031.586

   uca       ucc       ucm
   0.60      0.06      0.34

             Art        Cav        Mec
X(topt) =  9.185617  12.59281  40.33058
Y(topt) =  12.17004  28.23752  76.86983
```

Table 3.

```
Parameter fYa = -5

  Optimal Solution:

   objective function = -959.2858

   uca       ucc       ucm
   0.32      0.11      0.57

             Art        Cav        Mec
X(topt) =  8.382638  12.19132  39.83494
Y(topt) =  17.75569  26.98911  73.55618
```

4. Discussion

Palmquist (2016) writes that delaying operations are intended to buy time for one's own actions, inflict losses on an opponent, or limit the opponent's advance. However, as the reader of this article has seen, the purpose, or objective, can be generalized as follows: An objective function can and should be defined to simultaneously include several different "purposes," with different weights, synergy effects, and so on. This more general method of formulating and working with objective functions is also applied in this article. Palmquist (2016) continues with the statements that the brigade is a prerequisite for "achieving sufficient synergy effects," and that the brigade is the "optimal organizational form." This indicates that he, without explicitly analyzing and presenting combat and military organization as an optimization problem, nevertheless implicitly assumes that a brigade is the optimal solution to such an organizational and combat optimization problem. The author claims that a brigade "is assigned a delaying area that generally does not exceed 30 km in width and 50 km in depth," and that

the above-mentioned widths and depths should only be regarded as guidelines and must be adapted with regard to different conditions. Palmquist (2016) also writes that delaying operations are conducted with the support of delaying fieldworks and indirect fire. Delaying operations require that one's own combat effectiveness is preserved in order to enable endurance over time so that the mission can be accomplished. Delaying fieldworks provide significant combat effect and can be defended with limited resources. The need for fieldworks for survival must be taken into account.

The many principles and statements in the above text are logical and essential. However, no instructions on how to calculate the optimal levels of the many different measures are provided. It is therefore hoped that this new article provides the reader with useful suggestions and examples regarding methods for how to dimension the defense and its components, as well as how to manage the whole defense system in a rational manner.

5. Conclusions

This article presents new methods for optimizing delaying operations with brigades. The objective function takes into account all the costs of the war and its consequences. In advance, i.e. at the strategic level, the aim is to determine the optimal depth of defense, the optimal number of soldiers, and the optimal investments in field fortifications for protection as well as field works for attack. Over time, i.e. at the tactical level, the aim is to determine how the different battalions should allocate their total capacity, i.e. the use of time and thus also other resources. The optimal strategic and tactical decisions are influenced by the costs of various resources and measures. Examples of how different strategic and tactical decisions are affected by different conditions are presented.

6. Acknowledgements

Warm thanks to Lieutenant Colonel Dan Magnusson, who encouraged the author to analyze this subject; to Lieutenant Colonel Torbjörn Lundqvist and Captain Per-Axel Staflund, who allowed me to present the results at Umeå Garrison; and to Regimental Warrant Officer Class 1, Mats Wikberg, who provided me with current military regulations.

7. References

Bracken J. (1995). Lanchester models of Ardennes Campaign. Naval Research Logistics. 1995; 42: Pages 559–577.

Braun M. (1993). Differential Equations and Their Applications, Springer.

Chan P. (2016). The Lanchester Square Law: Its Implications for Force Structure and Force Preparation of Singapore's Operationally-Ready Soldiers, Pointer. Journal of the Singapore Armed Forces. 42(2).

Chen X, Qui J. (2014). Differential Game for a Class of Warfare Dynamic Systems with Reinforcement Based on Lanchester Equation. Abstract and Applied Analysis. 2014(8). doi: 10.1155/2014/837431.

Engel JH. (1954). A Verification of Lanchester's Law, Journal of the Operations Research Society of America, 1954; 2(2), Pages 163–171.

Eriksson, S. (2023). Reglemente Armé Taktik (R AR Taktik) version 2.0 Ä0. M7739-353050, Försvarsmakten, 107 85 Stockholm, 231 pages.

Eriksson, S., Palmquist, O. (2016). Handbok Markstrid – Bataljon (MSH Bat 2016), Försvarsmakten, 107 85 Stockholm, 184 pages.

Gyllenhaal, L. (2009). Elitförband i Norden, Fischer & Co, Stockholm, ISBN 978-91-85183-76-0, 178 pages.

Hung CY, Yang G, Deng P, et al. (2005). Fitting Lanchester's square law to the Ardennes Campaign. Journal of the Operations Research Society of America. 56: 942–946. doi: 10.1057/palgrave.jors.2601878

Hy MD, Vu MA, Nguyen NH, et al. (2020). Optimization in an asymmetric Lanchester (n, 1) model. Journal of Defense Modeling and Simulation: Applications, Methodology, Technology. 17(1): Pages 117–122. doi: 10.1177/1548512919828553.

Isaacs, R. (1965). Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization, John Wiley and Sons, Inc., New York, 384 pages.

Jörgensen, C. (2007). Great Battles, Parragon Books Ltd, Chartist House, Bath BA1 1HA, Great Britain, ISBN-13 9781405486583, 256 pages.

Kalloniatis AC, Hoek K, Zuparic M, Brede M. (2020). Optimising structure in a networked Lanchester Model for Fires and Manoeuvre in Warfare. Journal of the Operational Research Society. 72(8). Pages 1863–1878. doi:10.1080/01605682.2020.1745701

Kostic M, Jovanovic A. (2023). Lanchester's differential equations as operational command decision making tools. Serbian Journal of Management. 18(1). Pages 71–92.

Kress M, Caulkins JP, Feichtinger R, et al. (2018). Lanchester model for three-way combat, European Journal of Operational Research. 264(1). Pages 46–54. doi:10.1016/j.ejor.2017.07.026

Lanchester FW. (1916). Aircraft in Warfare: The Dawn of the Fourth Arm. Constable.

Lohmander, P. (2017). Applications and Mathematical Modeling in Operations Research, In: Cao BY. (ed) Fuzzy Information and Engineering and Decision. IWDS 2016. Advances in Intelligent Systems and Computing, vol 646. Springer, Cham, 2018 Print ISBN 978-3-319-66513-9, Online ISBN 978-3-319-66514-6, Pages 46-53, eBook Package: Engineering, https://doi.org/10.1007/978-3-319-66514-6_5

Lohmander, P. (2019a). Optimal decisions and expected values in two player zero sum games with diagonal game matrixes, - Explicit functions, general proofs and effects of parameter estimation errors, International Robotics & Automation Journal, Volume 5, Issue 5, 2019, pages 186-198. https://medcraveonline.com/IRATJ/IRATJ-05-00193.pdf

Lohmander, P. (2019b). Four central military decision problems, General methods and solutions, The Royal Swedish Academy of War Sciences Proceedings and Journal, 2/2019, pages 119-134. https://kkrva.se/hot/2019:2/09-lohmander-fyra-centrala-militara-beslutsproblem.pdf

Lohmander, P. (2023). Optimal Dynamic Control of Proxy War Arms Support, Automation, 2023, 4, pages 31-56. Publisher: MDPI Journals, Zürich, Switzerland. https://doi.org/10.3390/automation4010004 https://www.mdpi.com/2673-4052/4/1/4

Lohmander, P. (2024). Attrition coefficient estimations via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions, Journal of Statistics and Computer Science, Vol. 3, Issue 1, 2024, pages 51-78. Publisher: ARF India. https://www.arfjournals.com/image/catalog/Journals%20Papers/JSCS/2024/No%201%20(2024)/ART_4.pdf

Lohmander, P. (2025). Optimal Deployment, Mathematics and Systems Science, 2025, 3(1), 3055, pages 1–46. Publisher: Asia Pacific Academy of Science Pte. Ltd., Singapore. http://doi.org/10.54517/mss3055

Lystopadova V, Khalaim D. (2023) Application of Lanchester's mathematical laws in military strategy (Ukrainian). Osvita. Innovatyka. Praktyka. 2023; 11(8) Pages 44–50. doi: 10.31110/2616-650X-vol11i8-007

McCartney M. (2022). The solution of Lanchester's equations with inter-battle reinforcement strategies. Physica A: Statistical Mechanics and its Applications. 586: 126477. doi:10.1016/j.physa.2021.126477

Palmquist, O. (2016). Handbok Armé – Brigad 2016 (AH Brig 2016), Försvarsmakten, 107 85 Stockholm, 159 pages.

Sheeba PS, Ghose D. (2008). Optimal resource allocation in conflicts with the Lanchester linear law (2, 1) model. In: Proceedings of the 2008 American Control Conference; 11–13 June; Seattle, WA, USA. Pages 1806–1811.

Smedberg, M. (1994). Om Stridens Grunder, Page One Publishing AB, Stockholm, ISBN 91-7125-028-x, 255 pages.

Spradlin C, Spradlin G. (2007). Lanchester's equations in three dimensions. Computers and Mathematics with Applications. 2007(53). Pages 999–1011.

Stymfal MG. (2022). Revisiting Engel's verification of Lanchester's square law using battle of Iwo Jima data, Master thesis, Naval Postgraduate School, Monterey, USA.

Tam JH. (1998). Application of Lanchester combat model in the Ardennes campaign. Natural Resource Modeling. 11, Pages 95–116. doi: 10.1111/j.1939-7445.1998.tb00302.x

Washburn, A.R. (2002). Search and Detection, 4th Edition, Topics in Operations Research Series, Military Applications Society, Institute for Operations Research and the Management Sciences, MD 21090 USA, ISBN 1-877640-17-4, 149 pages.

Washburn, A.R. (2003). Two-Person Zero-Sum Games, 3rd Edition, Topics in Operations Research Series, Military Applications Society, Institute for Operations Research and the Management Sciences, MD 21090 USA, ISBN 1-877640-19-0, 125 pages.

Washburn, A.R., Kress, M. (2009). Combat Modeling, Springer, ISBN 978-1-4419-0789-9, 277 pages.

8. Appendix

Appendix 1.

Opt_bri

```
Rem Opt_bri_260221_1620.bas
Rem Peter Lohmander

Open "A_Opt_bri_Out.txt" For Output As #2
Screen _NewImage(400, 500, 256)

For P_Depth = 50 To 150 Step 10

   OptObj = -999999

   For Depth = 40 To 80 Step 5
      For FWA = 0 To 10
         For FWP = 0 To 10

            GoSub Parameters

            alpha = alpha_a * (50 / Depth) * (1 - .3 * FWP ^ .5)
            beta = beta_a * (Depth / 50) * (1 + .3 * FWA ^ .5)

            Lam = (alpha * beta) ^ .5
            bda = beta ^ .5 / alpha ^ .5

            For x0 = 1000 To 6000 Step 500
               A1 = ((x0 - v / beta) * bda - y0 + u / alpha) / (2 * bda)
               A2 = (y0 - u / alpha + bda * (x0 - v / beta)) / (2 * bda)

               GoSub Results

               If OptObj > Objective Then GoTo 100
               OptObj = Objective
               OptDepth = Depth
               Optx0 = x0
               OptFWA = FWA
               OptFWP = FWP
               Optalpha = alpha
               Optbeta = beta
               OptT = T
               OptxT = xT
```

```
                OptyT = yT
                100 Rem

            Next x0
          Next FWP
        Next FWA
    Next Depth

    Print #2, ""
    Print #2, ""
    Print #2, "P_Depth = "; P_Depth
    Print #2, ""

    Print #2, "   OptObj   = "; OptObj
    Print #2, "   Optx0    = "; Optx0
    Print #2, "   OptDepth = "; OptDepth
    Print #2, "   OptFWA   = "; OptFWA
    Print #2, "   OptFWP   = "; OptFWP
    Print #2, "   Optalpha = "; Optalpha
    Print #2, "   Optbeta  = "; Optbeta
    Print #2, "   OptT     = "; OptT
    Print #2, "   OptxT    = "; OptxT
    Print #2, "   OptyY    = "; OptyT

Next P_Depth

Close #2
End


Parameters:
alpha_a = 0.02
beta_a = 0.02
u = 50
v = 100
y0 = 3000
r = 0.01
Return


Results:
GoSub FinalT
T = TimeEnd
GoSub Xforce
xT = Xf
GoSub Yforce
yT = yF
```

```
Objective = -P_Depth * Depth - 1 * x0 - .0002 * x0 * x0 - 1 * u * T - 170 * FWP - 170 * FWA +
Exp(-r * T) * (xT - yT) - 10000 * ywins
Return


Xforce:
Xf = A1 * Exp(Lam * T) + A2 * Exp(-Lam * T) + v / beta
Return


Yforce:
yF = -bda * A1 * Exp(Lam * T) + bda * A2 * Exp(-Lam * T) + u / alpha
Return


FinalT:
TimeEnd = 1000
xwins = 0
ywins = 0
For T = 0 To 1000 Step 0.1
   GoSub Xforce
   GoSub Yforce
   If Xf < 0 Then TimeEnd = T
   If Xf < 0 Then ywins = 1
   If ywins = 1 Then Return
   If yF < 0 Then TimeEnd = T
   If yF < 0 Then xwins = 1
   If xwins = 1 Then Return
Next T
Return
```

Results from different versions of the software:

Table A11.

Results from optimization code "A_Opt_bri_260221_1620".

P_Depth	Opt_Obj	Opt_x0	Opt_Depth	Opt_FWA	Opt_FWP	Opt_Alpha	Opt_Beta	Opt_T	Opt_xT	Opt_yT
50	-8366	2500	80	5	1	0.00875	0.05347	43.7	4002	0
60	-9166	2500	80	5	1	0.00875	0.05347	43.7	4002	0
70	-9937	2500	75	5	2	0.007676	0.05012	47.3	4200	0
80	-10664	2500	70	6	3	0.006863	0.04858	48.8	4320	0
90	-11338	2500	65	6	4	0.006154	0.04511	54	4565	0
100	-11934	2500	60	7	5	0.005486	0.04305	57.1	4743	0
110	-12508	2500	50	7	10	0.001023	0.03587	65.9	5652	0
120	-13008	2500	50	7	10	0.001023	0.03587	65.9	5652	0
130	-13475	2500	45	8	10	0.00114	0.03327	73.8	6004	0
140	-13925	2500	45	8	10	0.00114	0.03327	73.8	6004	0
150	-14375	2500	45	8	10	0.00114	0.03327	73.8	6004	0

Table A12.

Results from optimization code "A_Opt_bri_260221_1648". This code is identical to the code "A_Opt_bri_260221_1620", with the following exception: There is a loop with different values of P_x0, and the parameter P_Depth is held constant at the value 100.

P_x0	Opt_Obj	Opt_x0	Opt_Depth	Opt_FWA	Opt_FWP	Opt_Alpha	Opt_Beta	Opt_T	Opt_xT	Opt_yT
0.5	-10443	3500	60	6	2	0.009596	0.04164	42.2	4919	0
0.6	-10793	3500	60	6	2	0.009596	0.04164	42.2	4919	0
0.7	-11101	3000	60	7	3	0.008006	0.04305	48.4	4726	0
0.8	-11402	3000	60	7	3	0.008006	0.04305	48.4	4726	0
0.9	-11702	3000	60	7	3	0.008006	0.04305	48.4	4726	0
1	-11964	2500	60	7	5	0.005486	0.04305	57.1	4743	0
1.1	-12177	2000	55	6	10	0.000933	0.03817	72.2	5456	0
1.2	-12377	2000	55	6	10	0.000933	0.03817	72.2	5456	0
1.3	-12554	1500	55	7	10	0.000933	0.03946	82.9	5446	0
1.4	-12704	1500	55	7	10	0.000933	0.03946	82.9	5446	0
1.5	-12854	1500	55	7	10	0.000933	0.03946	82.9	5446	0

Appendix 2.

```
Rem Opt_Brigade 260227_1444
Rem Peter Lohmander

Dim Xa(50), Xc(50), Xm(50)
Dim Ya(50), Yc(50), Ym(50)
Dim Optgraph(100)

Screen _NewImage(1800, 2000, 256)
Open "BrigOut.txt" For Output As #2

GoSub initial
GoSub increments
GoSub AttrCoef
GoSub Exponents
GoSub Objective_parameters
GoSub Decisions

For fYa = -15 To -5 Step 5
   Print ""
   Print #2, ""
   Print "  Parameter fYa = "; fYa
   Print #2, "  Parameter fYa = "; fYa

   Rem *** Optimization ***
   opt_value = -999999999
   opt_uca = 0
   opt_ucm = 0
   For uca = .01 To .99 Step 0.01
      ucm_max = 1 - uca
      For ucm = 0.01 To ucm_max Step .01
         ucc = 1 - uca - ucm
         For t = 0 To topt
            t2 = t + 1

            AttrXa = Ga * (vaa * Ya(t)) ^ raa * (vca * Yc(t)) ^ rca
            If AttrXa > Xa(t) Then AttrXa = Xa(t)
            Xa(t2) = Xa(t) + iXa - AttrXa

            AttrXc = Gc * (vac * Ya(t)) ^ rac * (vcc * Yc(t)) ^ rcc
            If AttrXc > Xc(t) Then AttrXc = Xc(t)
            Xc(t2) = Xc(t) + iXc - AttrXc

            AttrXm = Gm * (vam * Ya(t)) ^ ram * (vcm * Yc(t)) ^ rcm * (vmm * Ym(t)) ^ rmm
```

```
            If AttrXm > Xm(t) Then AttrXm = Xm(t)
            Xm(t2) = Xm(t) + iXm - AttrXm

            AttrYa = Ha * (uaa * Xa(t)) ^ saa * (uca * Xc(t)) ^ sca
            If AttrYa > Ya(t) Then AttrYa = Ya(t)
            Ya(t2) = Ya(t) + iYa - AttrYa

            AttrYc = Hc * (uac * Xa(t)) ^ sac * (ucc * Xc(t)) ^ scc
            If AttrYc > Yc(t) Then AttrYc = Yc(t)
            Yc(t2) = Yc(t) + iYc - AttrYc

            AttrYm = Hm * (uam * Xa(t)) ^ sam * (ucm * Xc(t)) ^ scm * (umm * Xm(t)) ^ smm
            If AttrYm > Ym(t) Then AttrYm = Ym(t)
            Ym(t2) = Ym(t) + iYm - AttrYm

        Next t
        GoSub Objective_function
        If f_value > opt_value Then GoSub New_Opt
    Next ucm
Next uca

Print ""
Print "    Optimal Solution:"
Print ""
Print "     objective function = "; opt_value
Print ""
Print "     uca      ucc      ucm"
Print Using "######.##"; opt_uca; opt_ucc; opt_ucm
Print ""
Print "             Art      Cav      Mec "
Print "   X(topt) = "; Xaopt; Xcopt; Xmopt
Print "   Y(topt) = "; Yaopt; Ycopt; Ymopt
Print " "

Print #2, ""
Print #2, "    Optimal Solution:"
Print #2, ""
Print #2, "     objective function = "; opt_value
Print #2, ""
Print #2, "     uca      ucc      ucm"
Print #2, Using "######.##"; opt_uca; opt_ucc; opt_ucm
Print #2, ""
Print #2, "             Art      Cav      Mec "
Print #2, "   X(topt) = "; Xaopt; Xcopt; Xmopt
Print #2, "   Y(topt) = "; Yaopt; Ycopt; Ymopt
Print #2, " "
```

```
Next fYa
Close #2
End

New_Opt:
opt_uca = uca
opt_ucc = ucc
opt_ucm = ucm
opt_value = f_value
Xaopt = Xa(topt)
Xcopt = Xc(topt)
Xmopt = Xm(topt)
Yaopt = Ya(topt)
Ycopt = Yc(topt)
Ymopt = Ym(topt)
Return

Objective_function:
f_value = fYa * Ya(topt) + fYc * Yc(topt) + fYm * Ym(topt)
Return

Objective_parameters:
topt = 10
fYa = -10
fYc = -5
fYm = -10
Return

initial:
Xa(0) = 6
Xc(0) = 6
Xm(0) = 18
Ya(0) = 2
Yc(0) = 2
Ym(0) = 6
Return

increments:
iXa = 1
iXc = 1
iXm = 3
iYa = 3
iYc = 3
iYm = 9
Return
```

AttrCoef:
Ga = .2
Gc = .1
Gm = .1
Ha = .5
Hc = .3
Hm = .3
Return

Exponents:
raa = .5
rca = 1 - raa
ram = 1 / 3
rcm = 1 / 3
rmm = 1 / 3
rac = .5
rcc = 1 - rac
saa = .5
sca = 1 - saa
sam = 1 / 3
scm = 1 / 3
smm = 1 / 3
sac = .5
scc = 1 - sac
Return

Decisions:
vaa = 1 / 3
vam = 1 / 3
vac = 1 / 3
vca = 1 / 3
vcm = 1 / 3
vcc = 1 / 3
vmm = 1
uaa = 1 / 3
uam = 1 / 3
uac = 1 / 3
uca = 1 / 3
ucm = 1 / 3
ucc = 1 / 3
umm = 1
Return