MATHEMATICAL APPENDIX

to

Optimal continuous cover forest management:
- Economic and environmental effects and legal considerations

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BIT's 5th Low Carbon Earth Summit

(LCES 2015 & ICE-2015)

Theme: "Take Actions for Rebuilding a Clean World"

September 24-26, 2015

Venue: Xi'an, China

This mathematical appendix includes derivations of most of the results presented in:

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- Economic and environmental effects and legal considerations”

by

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One and two dimensional optimization and comparative statistics in the volume and time interval dimensions

Peter Lohmander 2015-08-24

One dimensional optimization and comparative statistics in the time interval dimension

General Problem Definition

$$\max \; \pi = R(h) + \frac{P(v_1,t)Q(v_1,t)-c}{e^{rt}-1}$$

$$s.t.$$

$$h = v_0 - v_1$$

Obs: Generalized P(.), Q(.) and R(.). Assumption: r > 0.

Derivation of optimal General Principles

$$\frac{d\pi}{dt} = \frac{1}{\left(e^{rt}-1\right)^2}\left(\left(\frac{dP}{dt}Q + P\frac{dQ}{dt}\right)\left(e^{rt}-1\right) - \left(PQ-c\right)re^{rt}\right) = 0$$

$$\frac{d\pi}{dt} = \frac{e^{rt}}{\left(e^{rt}-1\right)^2}\left(\left(\frac{dP}{dt}Q + P\frac{dQ}{dt}\right)\left(1-e^{-rt}\right) - \left(PQ-c\right)r\right) = 0$$

$$k(.) = \frac{e^{rt}}{\left(e^{rt}-1\right)^2} > 0$$

$$M(.) = \left(\left(\frac{dP}{dt}Q + P\frac{dQ}{dt}\right)\left(1-e^{-rt}\right) - \left(PQ-c\right)r\right) = 0$$

$$\left(M(.) = 0\right) \Rightarrow$$

$$\left(M(.)=0\right)\Rightarrow$$

Optimal principles in the time dimension that can be given economic interpretations:

$$\frac{\dfrac{dP}{dt}Q+P\dfrac{dQ}{dt}}{\left(PQ-c\right)\dfrac{1}{1-e^{-rt}}}=r$$

$$\frac{\dfrac{dP}{dt}Q+P\dfrac{dQ}{dt}}{PQ+\dfrac{\left(-c+e^{-rt}PQ\right)}{1-e^{-rt}}}=r$$

$$\frac{dP}{dt}Q + P\frac{dQ}{dt} = r\left(\left(PQ - c\right)\frac{1}{1-e^{-rt}}\right)$$

$$\frac{dP}{dt}Q + P\frac{dQ}{dt} = r\left(PQ + \frac{\left(-c + e^{-rt}PQ\right)}{1-e^{-rt}}\right)$$

$$\frac{\frac{dP}{dt}Q + P\frac{dQ}{dt}}{\left(PQ - c\right)} = \frac{r}{1-e^{-rt}}$$

<u>**Comparative statistics analyses based on one dimensional optimization:**</u>

How is the optimal time interval affected if the parameter c marginally increases (ceteres paribus)?

The time intervals are optimized:

$$\frac{d\pi}{dt} = 0$$

If parameter changes take place, we still want the time interval to be optimal:

$$d\left(\frac{d\pi}{dt}\right) = 0$$

$$d\left(\frac{d\pi}{dt}\right)=\frac{d^2\pi}{dt^2}dt^*+\frac{d^2\pi}{dtdc}dc=0$$

$$\frac{d^2\pi}{dt^2}dt^*=-\frac{d^2\pi}{dtdc}dc$$

$$\frac{dt^*}{dc}=\frac{-\left(\dfrac{d^2\pi}{dtdc}\right)}{\left(\dfrac{d^2\pi}{dt^2}\right)}$$

A unique optimum is assumed in the time interval dimension. As a consequence:

$$\frac{d^2\pi}{dt^2}<0$$

$$\frac{d^2\pi}{dtdc} = k(.)r > 0$$

$$\frac{dt^*}{dc} = \frac{-\left(\dfrac{d^2\pi}{dtdc}\right)}{\left(\dfrac{d^2\pi}{dt^2}\right)} > 0$$

Conclusion:

The optimal time interval is a strictly increasing function of the parameter c.

Comparative statistics analyses based on one dimensional optimization:

How is the optimal time interval affected if the parameter r marginally increases (ceteres paribus)?

The time intervals are optimized:

$$\frac{d\pi}{dt} = 0$$

If parameter changes take place, we still want the time interval to be optimal:

$$d\left(\frac{d\pi}{dt}\right) = 0$$

$$d\left(\frac{d\pi}{dt}\right)=\frac{d^2\pi}{dt^2}dt^*+\frac{d^2\pi}{dtdr}dr=0$$

$$\frac{d^2\pi}{dt^2}dt^*=-\frac{d^2\pi}{dtdr}dr$$

$$\frac{dt^*}{dr}=\frac{-\left(\dfrac{d^2\pi}{dtdr}\right)}{\left(\dfrac{d^2\pi}{dt^2}\right)}$$

A unique optimum is assumed in the time interval dimension. As a consequence:

$$\frac{d^2\pi}{dt^2}<0$$

$$\frac{d^2\pi}{dtdr} = ?$$

$$\frac{d^2\pi}{dtdr} = \frac{dk}{dr}M + k\frac{dM}{dr}$$

$$\left(\frac{d\pi}{dt} = 0\right) \Rightarrow (M = 0) \Rightarrow \frac{d^2\pi}{dtdr} = k\frac{dM}{dr}$$

$$(k > 0) \Rightarrow \operatorname{sgn}\left(\frac{d^2\pi}{dtdr}\right) = \operatorname{sgn}\left(\frac{dM}{dr}\right)$$

$$M(.)=\left(\left(\frac{dP}{dt}Q+P\frac{dQ}{dt}\right)\left(1-e^{-rt}\right)-\left(PQ-c\right)r\right)=0$$

$$\frac{dM}{dr}=\left(\frac{dP}{dt}Q+P\frac{dQ}{dt}\right)te^{-rt}-\left(PQ-c\right)$$

$$r\frac{dM}{dr}=\left(\frac{dP}{dt}Q+P\frac{dQ}{dt}\right)rte^{-rt}-\left(PQ-c\right)r$$

$$r\frac{dM}{dr}-M=\left(\frac{dP}{dt}Q+P\frac{dQ}{dt}\right)\left(rte^{-rt}-\left(1-e^{-rt}\right)\right)$$

$$\bar{w} = \frac{dP}{dt} Q + P \frac{dQ}{dt} > 0$$

$$w = rte^{-rt} - \left(1 - e^{-rt}\right)$$

$$\operatorname{sgn}\left(r \frac{dM}{dr} - 0 \right) = \operatorname{sgn}\left(\bar{w} w \right) = \operatorname{sgn}(w)$$

The usual case:

$$\left(r>0\right)\Rightarrow \operatorname{sgn}\left(\frac{d^2\pi}{dtdr}\right)=\operatorname{sgn}\left(\frac{dM}{dr}\right)=\operatorname{sgn}\left(w\right)$$

$$w=rte^{-rt}-\left(1-e^{-rt}\right)$$

$$x=rt$$

$$w=xe^{-x}-1+e^{-x}$$

$$w(x)=\left(1+x\right)e^{-x}-1$$

$$w(0) = 0$$

$$\frac{dw}{dx} = 1e^{-x} + \left(1 + x\right)\left(-e^{-x}\right)$$

$$\frac{dw}{dx} = -xe^{-x}$$

$$\left(\frac{dw}{dx}\right)\bigg|_{x=0} = 0$$

$$\left(\frac{dw}{dx}\right)\bigg|_{x>0} < 0$$

$$x = rt > 0$$

$$w(x)\Big|_{x>0} < 0$$

$$\operatorname{sgn}\left(\frac{d^2\pi}{dtdr}\right) = \operatorname{sgn}\left(w\right) < 0$$

The unusual case:

$$\left(r<0\right) \Rightarrow \operatorname{sgn}\left(\frac{d^2\pi}{dtdr}\right)=\operatorname{sgn}\left(-\frac{dM}{dr}\right)=\operatorname{sgn}\left(-w\right)$$

$$x = rt < 0$$

$$w(x)\Big|_{x<0} > 0$$

$$\operatorname{sgn}\left(\frac{d^2\pi}{dtdr}\right)=\operatorname{sgn}\left(-w\right)<0$$

In both cases:

$$\frac{d^2\pi}{dtdr} < 0$$

$$\frac{dt^*}{dr} = \frac{-\left(\frac{d^2\pi}{dtdr}\right)}{\left(\frac{d^2\pi}{dt^2}\right)} < 0$$

Conclusion:

The optimal time interval is a strictly decreasing function of the parameter r.

<u>Comparative statistics analyses based on one dimensional optimization:</u>

How is the optimal time interval affected if the future prices marginally increase (ceteres paribus)?

In order to study the effects of prices on the optimal time interval, useful parameters are needed.

The problem is originally:

$$\max \pi = R(h) + \frac{P(v_1,t)Q(v_1,t) - c}{e^{rt} - 1}$$

Here, we replace this problem by this problem:

$$\max \pi = R(h) + \frac{pQ(v_1,t) - c}{e^{rt} - 1}$$

How is the optimal time interval affected if the parameter p marginally increases (ceteres paribus)?

The time intervals are optimized:

$$\frac{d\pi}{dt} = 0$$

If parameter changes take place, we still want the time interval to be optimal:

$$d\left(\frac{d\pi}{dt}\right) = 0$$

$$d\left(\frac{d\pi}{dt}\right)=\frac{d^2\pi}{dt^2}dt^*+\frac{d^2\pi}{dtdp}dp=0$$

$$\frac{d^2\pi}{dt^2}dt^*=-\frac{d^2\pi}{dtdp}dp$$

$$\frac{dt^*}{dp}=\frac{-\left(\dfrac{d^2\pi}{dtdp}\right)}{\left(\dfrac{d^2\pi}{dt^2}\right)}$$

A unique optimum is assumed in the time interval dimension. As a consequence:

$$\frac{d^2\pi}{dt^2} < 0$$

$$\frac{d\pi}{dt} = \frac{1}{\left(e^{rt}-1\right)^2}\left(p\frac{dQ}{dt}\left(e^{rt}-1\right)-\left(pQ-c\right)re^{rt} \right) = 0$$

$$\frac{d\pi}{dt} = \frac{e^{rt}}{\left(e^{rt}-1\right)^2}\left(p\frac{dQ}{dt}\left(1-e^{-rt}\right)-\left(pQ-c\right)r \right) = 0$$

$$k(.) = \frac{e^{rt}}{\left(e^{rt}-1\right)^2} > 0$$

$$M_1(.) = \left(p\frac{dQ}{dt}\left(1-e^{-rt}\right) - \left(pQ-c\right)r \right) = 0$$

$$\frac{d^2\pi}{dtdp} = k(.)\left(\frac{dQ}{dt}\left(1-e^{-rt}\right) - \left(Q\right)r \right)$$

$$p\frac{d^2\pi}{dtdp} = k(.)\left(p\frac{dQ}{dt}\left(1-e^{-rt}\right) - \left(pQ\right)r \right)$$

$$p\frac{d^2\pi}{dtdp}-\frac{d\pi}{dt}=k(.)\left(-cr\right)$$

$$\left((p>0)\wedge(cr>0)\wedge\left(\frac{d\pi}{dt}=0\right)\right)\Rightarrow\left(\frac{d^2\pi}{dtdp}<0\right)$$

$$\frac{dt^*}{dp}=\frac{-\left(\frac{d^2\pi}{dtdp}\right)}{\left(\frac{d^2\pi}{dt^2}\right)}<0$$

Conclusion:

The optimal time interval is a strictly decreasing function of the parameter p.

One dimensional optimization and comparative statistics in the volume dimension

General Problem Definition

$$\max \pi = R(h) + \frac{P(v_1, t)Q(v_1, t) - c}{e^{rt} - 1}$$

$$s.t.$$

$$h = v_0 - v_1$$

Obs: Generalized P(.), Q(.) and R(.). t > 0. Assumption: r > 0.

$$\frac{d\pi}{dv_1} = \frac{dR}{dh}\frac{dh}{dv_1} + \frac{1}{e^{rt}-1}\left(\frac{dP}{dv_1}Q(.) + P\frac{dQ}{dv_1}\right) = 0$$

$$\frac{dh}{dv_1} = -1$$

The first order optimum condition implies:

$$\frac{dR}{dh} = \frac{1}{e^{rt}-1}\left(\frac{dP}{dv_1}Q(.) + P\frac{dQ}{dv_1}\right)$$

Assumption: A unique maximum in the volume dimension:

$$\frac{d^2\pi}{dv_1{}^2} < 0$$

Assumption:

$$\frac{dR}{dh} > 0$$

$$\left(\frac{dR}{dh} > 0 \wedge \frac{d\pi}{dv_1} = 0 \wedge rt > 0 \right) \Rightarrow \left(\frac{dP}{dv_1} Q(.) + P \frac{dQ}{dv_1} > 0 \right)$$

Assumption:

$$\frac{d^2\pi}{dv_1 dr} = \frac{-e^{rt}}{e^{rt} - 1} \left(\frac{dP}{dv_1} Q(.) + P \frac{dQ}{dv_1} \right) < 0$$

Comparative statistics analyses based on one dimensional optimization:

How is the optimal volume affected if the parameter r marginally increases (ceteres paribus)?

The volume is optimized:

$$\frac{d\pi}{dv_1} = 0$$

If parameter changes take place, we still want the volume to be optimal:

$$d\left(\frac{d\pi}{dv_1}\right) = 0$$

$$d\left(\frac{d\pi}{dv_1}\right)=\frac{d^2\pi}{dv_1{}^2}dv_1{}^* + \frac{d^2\pi}{dv_1 dr}dr = 0$$

$$\frac{d^2\pi}{dv_1{}^2}dv_1{}^* + \frac{d^2\pi}{dv_1 dr}dr = 0$$

$$\frac{d^2\pi}{dv_1{}^2}dv_1{}^* = -\frac{d^2\pi}{dv_1 dr}dr$$

$$\frac{dv_1{}^*}{dr} = -\frac{\left(\dfrac{d^2\pi}{dv_1 dr}\right)}{\left(\dfrac{d^2\pi}{dv_1{}^2}\right)} < 0$$

Conclusion:

The optimal volume is a strictly decreasing function of the parameter r.

<u>Comparative statistics analyses based on one dimensional optimization:</u>

How is the optimal volume affected if the future prices marginally increase (ceteres paribus)?

In order to study the effects of prices on the optimal time interval, useful parameters are needed.

The problem is originally:

$$\max \pi = R(h) + \frac{P(v_1,t)Q(v_1,t) - c}{e^{rt} - 1}$$

Here, we replace this problem by this problem:

$$\max \pi = R(h) + \frac{pQ(v_1,t) - c}{e^{rt} - 1}$$

$$\frac{d\pi}{dv_1} = \frac{dR}{dh}\frac{dh}{dv_1} + \left(\frac{1}{e^{rt}-1}\right)p\frac{dQ}{dv_1} = 0$$

$$\frac{d^2\pi}{dv_1 dp} = \left(\frac{1}{e^{rt}-1}\right)\frac{dQ}{dv_1}$$

The first order optimum condition implies:

$$\frac{dR}{dh} = \left(\frac{1}{e^{rt}-1}\right)p\frac{dQ}{dv_1}$$

Assumption: A unique maximum in the volume dimension:

$$\frac{d^2\pi}{dv_1{}^2} < 0$$

Assumption:

$$\frac{dR}{dh} > 0$$

$$\left(\frac{dR}{dh} > 0 \wedge \frac{d\pi}{dv_1} = 0 \wedge rt > 0 \wedge p > 0 \right) \Rightarrow \left(\left(\frac{1}{e^{rt} - 1} \right) \frac{dQ}{dv_1} > 0 \right)$$

$$\frac{d^2\pi}{dv_1 dp} = \left(\frac{1}{e^{rt} - 1} \right) \frac{dQ}{dv_1} > 0$$

$$d\left(\frac{d\pi}{dv_1} \right) = \frac{d^2\pi}{dv_1{}^2} dv_1{}^* + \frac{d^2\pi}{dv_1 dp} dp = 0$$

$$\frac{d^2\pi}{dv_1{}^2}dv_1{}^* + \frac{d^2\pi}{dv_1 dp}dp = 0$$

$$\frac{d^2\pi}{dv_1{}^2}dv_1{}^* = -\frac{d^2\pi}{dv_1 dp}dp$$

$$\frac{dv_1{}^*}{dp} = -\frac{\left(\dfrac{d^2\pi}{dv_1 dp}\right)}{\left(\dfrac{d^2\pi}{dv_1{}^2}\right)} > 0$$

Conclusion:

The optimal volume is a strictly increasing function of the parameter p.

Comparative statistics analyses based on one dimensional optimization:

How is the optimal volume affected if the parameter c marginally increases (ceteres paribus)?

$$\frac{d^2\pi}{dv_1{}^2} dv_1{}^* + \frac{d^2\pi}{dv_1 dc} dc = 0$$

$$\frac{dv_1{}^*}{dc} = -\frac{\left(\dfrac{d^2\pi}{dv_1 dc}\right)}{\left(\dfrac{d^2\pi}{dv_1{}^2}\right)} = -\frac{0}{\left(\dfrac{d^2\pi}{dv_1{}^2}\right)} = 0$$

Conclusion:

The optimal volume is not affected by changes of the parameter c.

Observation:

If the volume is constrained (by law or something else), we may study the effects of the volume constraint on the optimal time interval, via one dimensional optimization and

comparative statics using $\frac{d^2\pi}{dv_1 dt}$.

Also, since $\frac{d^2\pi}{dv_1 dt} \equiv \frac{d^2\pi}{dt dv_1}$, we may perform the corresponding analysis of optimal volume based on constrained t.

In both cases, it is necessary to know somehing about $\frac{d^2\pi}{dv_1 dt}$.

Below, we will also use $\frac{d^2\pi}{dv_1 dt}$ in another way, in two dimensional optimization of the time interval and the volume.

We will soon discover that $\frac{d^2\pi}{dv_1 dt} < 0$.

In ”one dimensional optimization” and comparative statics analysis, we come to the following conclusions:

$$\frac{dt^*}{dv_1} = \frac{-\left(\frac{d^2\pi}{dt dv_1}\right)}{\left(\frac{d^2\pi}{dt^2}\right)} < 0$$

and

$$\frac{dv_1^*}{dt} = \frac{-\left(\frac{d^2\pi}{dv_1 dt}\right)}{\left(\frac{d^2\pi}{dv_1^2}\right)} < 0$$

Two dimensional optimization and comparative statistics in the volume and time interval dimensions

General Problem Definition

$$\max \pi = R(h) + \frac{pQ(v_1, t) - c}{e^{rt} - 1}$$

$$s.t.$$

$$h = v_0 - v_1$$

The first order optimum conditions:

$$\begin{cases} \dfrac{d\pi}{dv_1} = 0 \\ \dfrac{d\pi}{dt} = 0 \end{cases}$$

$$\begin{cases} \dfrac{dR}{dh}\dfrac{dh}{dv_1} + \left(\dfrac{1}{e^{rt}-1}\right) p \dfrac{dQ}{dv_1} = 0 \\ \dfrac{e^{rt}}{\left(e^{rt}-1\right)^2}\left(p\dfrac{dQ}{dt}\left(1-e^{-rt}\right) - \left(pQ-c\right)r \right) = 0 \end{cases}$$

Assumption: A unique maximum exists. The following conditions hold:

$$\left|\frac{d^2\pi}{dv_1^{\,2}}\right| < 0, \quad \left|\frac{d^2\pi}{dt^2}\right| < 0, \quad \begin{vmatrix} \dfrac{d^2\pi}{dv_1^{\,2}} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dt dv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix} > 0$$

For a unique maximum, it is sufficient that:

$$\left|\frac{d^2\pi}{dv_1^{\,2}}\right| < 0, \quad \begin{vmatrix} \dfrac{d^2\pi}{dv_1^{\,2}} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dt dv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix} > 0$$

In the analysis, it will turn out that it is important to know something about

$$\frac{d^2\pi}{dv_1 dt} \equiv \frac{d^2\pi}{dt dv_1}$$, at least the sign.

$$\left(|D| = \begin{vmatrix} \dfrac{d^2\pi}{dv_1^{\;2}} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dt dv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix} > 0 \right) \Rightarrow \frac{d^2\pi}{dv_1^{\;2}} \frac{d^2\pi}{dt^2} - \frac{d^2\pi}{dt dv_1} \frac{d^2\pi}{dv_1 dt} > 0$$

$$\frac{d^2\pi}{dv_1^{\,2}}\frac{d^2\pi}{dt^2} > \frac{d^2\pi}{dtdv_1}\frac{d^2\pi}{dv_1dt} = \left(\frac{d^2\pi}{dv_1dt}\right)^2$$

Observation:

$\left(\frac{d^2\pi}{dv_1dt}\right)^2$ is usually small in relation to $\frac{d^2\pi}{dv_1^{\,2}}\frac{d^2\pi}{dt^2}$. $\frac{d^2\pi}{dv_1dt}$ is however a function of several parameters and decision variables, which makes it complicated to give general and explicit functions describing the signs and absolute values of $\frac{d^2\pi}{dv_1dt}$. In any case, the sign of $\frac{d^2\pi}{dv_1dt}$ will be determined.

Determination of the sign of $\frac{d^2\pi}{dv_1 dt}$ **:**

$$\frac{d\pi}{dv_1} = -\frac{dR}{dh} + \left(\frac{1}{e^{rt}-1}\right) p \frac{dQ}{dv_1} = 0$$

$$\frac{d^2\pi}{dv_1 dt} = \frac{-re^{rt}}{\left(e^{rt}-1\right)^2} p \frac{dQ}{dv_1} + \left(\frac{1}{e^{rt}-1}\right) p \frac{d^2Q}{dv_1 dt}$$

$$\frac{d^2\pi}{dv_1 dt} = \frac{p}{e^{rt}-1}\left(\frac{d^2Q}{dtdv_1} - \left(\frac{r}{1-e^{-rt}}\right)\frac{dQ}{dv_1}\right)$$

$$\frac{d\pi}{dt} = \frac{1}{\left(e^{rt}-1\right)^2}\left(p\frac{dQ}{dt}\left(e^{rt}-1\right) - \left(pQ-c\right)re^{rt}\right) = 0$$

$$\frac{d^2\pi}{dtdv_1} = \frac{p}{e^{rt}-1}\left(\frac{d^2Q}{dtdv_1} - \left(\frac{r}{1-e^{-rt}}\right)\frac{dQ}{dv_1}\right)$$

Observation:

$$\frac{d^2\pi}{dv_1 dt} = \frac{d^2\pi}{dt dv_1} = \frac{p}{e^{rt}-1}\left(\frac{d^2Q}{dt dv_1} - \left(\frac{r}{1-e^{-rt}}\right)\frac{dQ}{dv_1}\right)$$

$$\operatorname{sgn}\left(\frac{d^2\pi}{dv_1 dt}\right) = \operatorname{sgn}(\theta)$$

$$\theta = \frac{d^2Q}{dv_1 dt} - \left(\frac{r}{1-e^{-rt}}\right)\frac{dQ}{dv_1}$$

With relevant growth functions:

$$\int_0^t \frac{d^2Q}{dv_1dt}ds > \frac{d^2Q}{dv_1dt}t$$

$$\int_0^t \frac{d^2Q}{dv_1dt}ds = \frac{dQ}{dv_1}$$

$$\frac{d^2Q}{dv_1dt}t < \frac{dQ}{dv_1}$$

$$\frac{d^2Q}{dv_1dt} = \frac{\left(\dfrac{dQ}{dv_1}\right)}{t} - k, \quad k > 0$$

$$\theta = \frac{d^2Q}{dv_1 dt} - \left(\frac{r}{1-e^{-rt}} \right) \frac{dQ}{dv_1}$$

$$\theta = \frac{\left(\dfrac{dQ}{dv_1} \right)}{t} - k - \left(\frac{r}{1-e^{-rt}} \right) \frac{dQ}{dv_1}$$

$$\theta = \frac{dQ}{dv_1} \left(\frac{1}{t} - k - \frac{r}{1-e^{-rt}} \right)$$

Observation:

$$\left(\frac{dQ}{dv_1} > 0 \right) \Rightarrow \operatorname{sgn}(\theta) = \operatorname{sgn}\left(\frac{1}{t} - k - \frac{r}{1-e^{-rt}} \right)$$

Let $w = \dfrac{r}{1-e^{-rt}}$

W(r=0) = ? Observation: Division by zero is not allowed.

$$\lim_{r\to 0}\left(\frac{r}{1-e^{-rt}}\right) = \frac{\left(\dfrac{dr}{dr}\right)}{\left(\dfrac{d\left(1-e^{-rt}\right)}{dr}\right)} = \frac{1}{te^{-rt}} = \frac{1}{t}$$

$$\lim_{r\to 0}\theta = \frac{dQ}{dv_1}\left(\frac{1}{t} - k - \frac{1}{t}\right) = -k\,\frac{dQ}{dv_1} < 0$$

$$\theta = \frac{dQ}{dv_1}\left(\frac{1}{t} - k - \frac{r}{1-e^{-rt}}\right)$$

$$\frac{d\theta}{dr} = \frac{dQ}{dv_1}\left(\frac{-\left(1\left(1-e^{-rt}\right) - r\left(te^{-rt}\right)\right)}{\left(1-e^{-rt}\right)^2}\right)$$

$$\frac{d\theta}{dr} = \frac{dQ}{dv_1}\frac{-1}{\left(1-e^{-rt}\right)^2}\left(1-e^{-rt} - rte^{-rt}\right)$$

Let

$$\phi = \left(1 - e^{-rt}(1 + rt)\right)$$

Observation:

$$\operatorname{sgn}\left(\frac{d\theta}{dr}\right) = \operatorname{sgn}\left(\phi\right)$$

$$\phi(r = 0) = 0$$

$$\frac{d\phi}{dr} = te^{-rt}(1 + rt) - e^{-rt}t$$

$$\frac{d\phi}{dr} = rt^2 e^{-rt}$$

$$\left(r > 0 \wedge t > 0\right) \Rightarrow \frac{d\phi}{dr} > 0$$

$$\left(r > 0 \wedge t > 0\right) \Rightarrow \left(\phi > 0\right)$$

$$\operatorname{sgn}\left(\frac{d\theta}{dr}\right) = \operatorname{sgn}\left(-\phi\right)$$

$$\frac{d\theta}{dr} < 0$$

$$\left(\lim_{r \to 0} \theta < 0 \wedge \frac{d\theta}{dr} < 0 \right) \Rightarrow \left(\theta < 0 \right)$$

$$\operatorname{sgn}\left(\frac{d^2\pi}{dv_1 dt} \right) = \operatorname{sgn}\left(\theta \right)$$

$$\frac{d^2\pi}{dv_1 dt} < 0$$

Comparative statics analysis based on two dimensional optimization:

$$\begin{bmatrix} \frac{d^2\pi}{dv_1^{\,2}} & \frac{d^2\pi}{dv_1 dt} \\ \frac{d^2\pi}{dtdv_1} & \frac{d^2\pi}{dt^2} \end{bmatrix} \begin{bmatrix} dv_1^{\,*} \\ dt^* \end{bmatrix} = \begin{bmatrix} -\frac{d^2\pi}{dv_1 dc}dc - \frac{d^2\pi}{dv_1 dp}dp - \frac{d^2\pi}{dv_1 dr}dr \\ -\frac{d^2\pi}{dtdc}dc - \frac{d^2\pi}{dtdp}dp - \frac{d^2\pi}{dtdr}dr \end{bmatrix}$$

First, we study the effects of changes in c, when $dp = dr = 0$.

$$\frac{d^2\pi}{dv_1 dc} = 0$$

$$\frac{d^2\pi}{dt dc} = \frac{re^{rt}}{\left(e^{rt} - 1\right)^2} > 0$$

$$\begin{bmatrix} \frac{d^2\pi}{dv_1^{\ 2}} & \frac{d^2\pi}{dv_1 dt} \\ \frac{d^2\pi}{dt dv_1} & \frac{d^2\pi}{dt^2} \end{bmatrix} \begin{bmatrix} dv_1^{\ *} \\ dt^* \end{bmatrix} = \begin{bmatrix} -\frac{d^2\pi}{dv_1 dc}dc \\ -\frac{d^2\pi}{dt dc}dc \end{bmatrix}$$

$$\frac{dv_1^{\ *}}{dc} = \frac{\begin{vmatrix} -\frac{d^2\pi}{dv_1 dc} & \frac{d^2\pi}{dv_1 dt} \\ -\frac{d^2\pi}{dt dc} & \frac{d^2\pi}{dt^2} \end{vmatrix}}{\begin{vmatrix} \frac{d^2\pi}{dv_1^{\ 2}} & \frac{d^2\pi}{dv_1 dt} \\ \frac{d^2\pi}{dt dv_1} & \frac{d^2\pi}{dt^2} \end{vmatrix}} = \frac{\begin{vmatrix} -\frac{d^2\pi}{dv_1 dc} & \frac{d^2\pi}{dv_1 dt} \\ -\frac{d^2\pi}{dt dc} & \frac{d^2\pi}{dt^2} \end{vmatrix}}{|D|}$$

$$\frac{dv_1^*}{dc} = \frac{\begin{vmatrix} 0 & \dfrac{d^2\pi}{dv_1 dt} \\ -\dfrac{d^2\pi}{dtdc} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{|D|} = \frac{\dfrac{d^2\pi}{dtdc}\dfrac{d^2\pi}{dv_1 dt}}{|D|}$$

$$\left(\frac{d^2\pi}{dtdc} > 0 \wedge |D| > 0\right) \Rightarrow \operatorname{sgn}\left(\frac{dv_1^*}{dc}\right) = \operatorname{sgn}\left(\frac{d^2\pi}{dv_1 dt}\right)$$

Observation:

$\dfrac{d^2\pi}{dv_1 dt} < 0$. As a consequence, $\dfrac{dv_1^*}{dc} < 0$.

$$\frac{dt^*}{dc} = \frac{\begin{vmatrix} \dfrac{d^2\pi}{dv_1^{\;2}} & -\dfrac{d^2\pi}{dv_1 dc} \\ \dfrac{d^2\pi}{dtdv_1} & -\dfrac{d^2\pi}{dtdc} \end{vmatrix}}{\begin{vmatrix} \dfrac{d^2\pi}{dv_1^{\;2}} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dtdv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}} = \frac{\begin{vmatrix} \dfrac{d^2\pi}{dv_1^{\;2}} & 0 \\ \dfrac{d^2\pi}{dtdv_1} & -\dfrac{d^2\pi}{dtdc} \end{vmatrix}}{|D|} = \frac{-\dfrac{d^2\pi}{dv_1^{\;2}}\dfrac{d^2\pi}{dtdc}}{|D|} > 0$$

Second, we study the effects of changes in p, when $dc = dr = 0$.

$$\frac{d^2\pi}{dv_1 dp} = \left(\frac{1}{e^{rt}-1}\right)\frac{dQ}{dv_1} > 0$$

$$\frac{d^2\pi}{dtdp} = \frac{1}{\left(e^{rt}-1\right)^2}\left(\frac{dQ}{dt}\left(e^{rt}-1\right) - \left(Q\right)re^{rt}\right)$$

$$p\frac{d^2\pi}{dtdp} = \frac{1}{\left(e^{rt}-1\right)^2}\left(p\frac{dQ}{dt}\left(e^{rt}-1\right) - \left(pQ\right)re^{rt}\right)$$

$$p\frac{d^2\pi}{dtdp}-\frac{d\pi}{dt}=\frac{1}{\left(e^{rt}-1\right)^2}\left(-cre^{rt}\right)<0$$

$$\left(p>0\wedge\frac{d\pi}{dt}=0\right)\Rightarrow\frac{d^2\pi}{dtdp}<0$$

$$\begin{bmatrix}\dfrac{d^2\pi}{dv_1{}^2} & \dfrac{d^2\pi}{dv_1dt}\\ \dfrac{d^2\pi}{dtdv_1} & \dfrac{d^2\pi}{dt^2}\end{bmatrix}\begin{bmatrix}dv_1{}^*\\ dt^*\end{bmatrix}=\begin{bmatrix}-\dfrac{d^2\pi}{dv_1dp}dp\\ -\dfrac{d^2\pi}{dtdp}dp\end{bmatrix}$$

$$\frac{dv_1^*}{dp} = \frac{\begin{vmatrix} -\dfrac{d^2\pi}{dv_1dp} & \dfrac{d^2\pi}{dv_1dt} \\ -\dfrac{d^2\pi}{dtdp} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{\begin{vmatrix} \dfrac{d^2\pi}{dv_1^2} & \dfrac{d^2\pi}{dv_1dt} \\ \dfrac{d^2\pi}{dtdv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}} = \frac{\begin{vmatrix} -\dfrac{d^2\pi}{dv_1dp} & \dfrac{d^2\pi}{dv_1dt} \\ -\dfrac{d^2\pi}{dtdp} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{|D|}$$

$$\frac{dv_1^*}{dp} = \frac{\begin{vmatrix} -\dfrac{d^2\pi}{dv_1dp} & \dfrac{d^2\pi}{dv_1dt} \\ -\dfrac{d^2\pi}{dtdp} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{|D|} = \frac{-\dfrac{d^2\pi}{dv_1dp}\dfrac{d^2\pi}{dt^2} + \dfrac{d^2\pi}{dtdp}\dfrac{d^2\pi}{dv_1dt}}{|D|} > 0$$

$$\frac{dt^*}{dp} = \frac{\begin{vmatrix} \dfrac{d^2\pi}{dv_1{}^2} & -\dfrac{d^2\pi}{dv_1 dp} \\ \dfrac{d^2\pi}{dtdv_1} & -\dfrac{d^2\pi}{dtdp} \end{vmatrix}}{|D|} = \frac{-\dfrac{d^2\pi}{dv_1{}^2}\dfrac{d^2\pi}{dtdp} + \dfrac{d^2\pi}{dtdv_1}\dfrac{d^2\pi}{dv_1 dp}}{|D|} < 0$$

Third, we study the effects of changes in r, when $dc = dp = 0$.

$$\frac{d^2\pi}{dv_1 dr} = \frac{-pte^{rt}\dfrac{dQ}{dv_1}}{\left(e^{rt}-1\right)^2} < 0$$

We have already determined that that:

$$\frac{d^2\pi}{dtdr} < 0$$

$$\begin{bmatrix} \dfrac{d^2\pi}{dv_1^{\,2}} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dtdv_1} & \dfrac{d^2\pi}{dt^2} \end{bmatrix} \begin{bmatrix} dv_1^{\,*} \\ dt^* \end{bmatrix} = \begin{bmatrix} -\dfrac{d^2\pi}{dv_1 dr}dr \\ -\dfrac{d^2\pi}{dtdr}dr \end{bmatrix}$$

$$\frac{dv_1^*}{dr} = \frac{\begin{vmatrix} -\dfrac{d^2\pi}{dv_1 dr} & \dfrac{d^2\pi}{dv_1 dt} \\ -\dfrac{d^2\pi}{dtdr} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{\begin{vmatrix} \dfrac{d^2\pi}{dv_1^2} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dtdv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}} = \frac{\begin{vmatrix} -\dfrac{d^2\pi}{dv_1 dr} & \dfrac{d^2\pi}{dv_1 dt} \\ -\dfrac{d^2\pi}{dtdr} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{|D|}$$

$$\frac{dv_1^*}{dr} = \frac{\begin{vmatrix} -\dfrac{d^2\pi}{dv_1 dr} & \dfrac{d^2\pi}{dv_1 dt} \\ -\dfrac{d^2\pi}{dtdr} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{|D|} = \frac{-\dfrac{d^2\pi}{dv_1 dr}\dfrac{d^2\pi}{dt^2} + \dfrac{d^2\pi}{dtdr}\dfrac{d^2\pi}{dv_1 dt}}{|D|}$$

$\dfrac{d^2\pi}{dv_1 dt}$ is negative but has a low absolute value. As a result: $\dfrac{dv_1^*}{dr} < 0$

$$\frac{dt^*}{dr} = \frac{\begin{vmatrix} \frac{d^2\pi}{dv_1^{\,2}} & -\frac{d^2\pi}{dv_1 dr} \\ \frac{d^2\pi}{dtdv_1} & -\frac{d^2\pi}{dtdr} \end{vmatrix}}{\begin{vmatrix} \frac{d^2\pi}{dv_1^{\,2}} & \frac{d^2\pi}{dv_1 dt} \\ \frac{d^2\pi}{dtdv_1} & \frac{d^2\pi}{dt^2} \end{vmatrix}} = \frac{-\frac{d^2\pi}{dv_1^{\,2}}\frac{d^2\pi}{dtdr} + \frac{d^2\pi}{dtdv_1}\frac{d^2\pi}{dv_1 dr}}{|D|}$$

$\frac{d^2\pi}{dv_1 dt}$ is negative but has a low absolute value. As a result: $\frac{dt^*}{dr} < 0$

Optimal continuous cover forest management

First, we study:

One dimensional optimization in the time interval dimension (of relevance when the stock level after harvest is determined by law or can not be determined for some other reason)

$$\max \pi = R(h) + \frac{P(v_1, t)Q(v_1, t) - c}{e^{rt} - 1}$$

$$s.t.$$

$$h = v_0 - v_1$$

$$\frac{d\pi}{dt}=\frac{e^{rt}}{\left(e^{rt}-1\right)^2}\left(\left(\frac{dP}{dt}Q+P\frac{dQ}{dt}\right)\left(1-e^{-rt}\right)-\left(PQ-c\right)r\right)=0$$

$$\left(\frac{dP}{dt}Q+P\frac{dQ}{dt}\right)\left(1-e^{-rt}\right)-\left(PQ-c\right)r=0$$

$\left(M(.)=0\right)\Rightarrow$

Optimal principle in the time dimension:

$$\frac{\dfrac{dP}{dt}Q+P\dfrac{dQ}{dt}}{PQ+\dfrac{\left(-c+e^{-rt}PQ\right)}{1-e^{-rt}}}=r$$

How is the optimal time interval affected if the parameter c marginally increases (ceteres paribus)?

$$\frac{d\pi}{dt} = 0$$

$$d\left(\frac{d\pi}{dt}\right) = \frac{d^2\pi}{dt^2}dt^* + \frac{d^2\pi}{dtdc}dc = 0$$

$$\frac{dt^*}{dc} = \frac{-\left(\frac{d^2\pi}{dtdc}\right)}{\left(\frac{d^2\pi}{dt^2}\right)}$$

A unique optimum is assumed in the time interval dimension

$$\frac{d^2\pi}{dt^2} < 0$$

$$\frac{d^2\pi}{dtdc} = k(.)r > 0$$

$$\frac{dt^*}{dc} = \frac{-\left(\dfrac{d^2\pi}{dtdc}\right)}{\left(\dfrac{d^2\pi}{dt^2}\right)} > 0$$

Conclusion:

The optimal time interval is a strictly increasing function of c.

How is the optimal time interval affected if the parameter r marginally increases (ceteres paribus)?

$$\frac{dt^*}{dr} = \frac{-\left(\dfrac{d^2\pi}{dtdr}\right)}{\left(\dfrac{d^2\pi}{dt^2}\right)} < 0$$

Conclusion:

The optimal time interval is a strictly decreasing function of the parameter r.

How is the optimal time interval affected if the future prices marginally increase (ceteres paribus)?

$$\max \pi = R(h) + \frac{pQ(v_1, t) - c}{e^{rt} - 1}$$

$$\frac{dt^*}{dp} = \frac{-\left(\dfrac{d^2\pi}{dtdp}\right)}{\left(\dfrac{d^2\pi}{dt^2}\right)} < 0$$

Conclusion:

The optimal time interval is a strictly decreasing function of the parameter p.

Optimal continuous cover forest management

Now, we study:

One dimensional optimization in the volume dimension (of relevance when the time interval is determined by law or can not be determined for some other reason)

$$\max \pi = R(h) + \frac{P(v_1, t)Q(v_1, t) - c}{e^{rt} - 1}$$

$$s.t.$$

$$h = v_0 - v_1$$

$$\frac{d\pi}{dv_1} = \frac{dR}{dh}\frac{dh}{dv_1} + \frac{1}{e^{rt}-1}\left(\frac{dP}{dv_1}Q(.) + P\frac{dQ}{dv_1}\right) = 0$$

$$\frac{dh}{dv_1} = -1$$

Optimal principle in the volume dimension:

$$\frac{dR}{dh} = \frac{1}{e^{rt}-1}\left(\frac{dP}{dv_1}Q(.) + P\frac{dQ}{dv_1}\right)$$

How is the optimal volume affected if one parameter marginally increases (ceteres paribus)?

- The optimal volume is not affected by changes of c.

- The optimal volume is a strictly decreasing function of r.

- The optimal volume is a strictly increasing function of p.

Observation:

- If the volume is constrained (by law or something else), we may study the effects of the volume constraint on the optimal time interval, via one dimensional optimization.

$$\frac{d^2\pi}{dv_1 dt} < 0$$

$$\frac{dt^*}{dv_1} = \frac{-\left(\dfrac{d^2\pi}{dt dv_1}\right)}{\left(\dfrac{d^2\pi}{dt^2}\right)} < 0$$

Observation (extended):

- If the time interval is constrained (by law or something else), we may study the effects of the time interval constraint on the volume, via one dimensional optimization.

$$\frac{d^2\pi}{dv_1 dt} < 0$$

$$\frac{dv_1^*}{dt} = \frac{-\left(\frac{d^2\pi}{dv_1 dt}\right)}{\left(\frac{d^2\pi}{dv_1^2}\right)} < 0$$

Optimal continuous cover forest management

Now, we study:

Two dimensional optimization in the volume AND time interval dimensions

$$\max \pi = R(h) + \frac{pQ(v_1, t) - c}{e^{rt} - 1}$$

$$s.t.$$

$$h = v_0 - v_1$$

The first order optimum conditions:

$$\begin{cases} \dfrac{d\pi}{dv_1} = 0 \\ \dfrac{d\pi}{dt} = 0 \end{cases}$$

$$\begin{cases} \dfrac{dR}{dh}\dfrac{dh}{dv_1} + \left(\dfrac{1}{e^{rt} - 1}\right) p \dfrac{dQ}{dv_1} = 0 \\ \dfrac{e^{rt}}{\left(e^{rt} - 1\right)^2}\left(p\dfrac{dQ}{dt}\left(1 - e^{-rt}\right) - \left(pQ - c\right) r \right) = 0 \end{cases}$$

Assumption: A unique maximum exists. The following conditions hold:

$$\left|\frac{d^2\pi}{dv_1{}^2}\right| < 0, \qquad \left|\frac{d^2\pi}{dt^2}\right| < 0, \qquad \begin{vmatrix} \dfrac{d^2\pi}{dv_1{}^2} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dt dv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix} > 0$$

$$\left(|D| = \begin{vmatrix} \dfrac{d^2\pi}{dv_1{}^2} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dt dv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix} > 0 \right) \Rightarrow \frac{d^2\pi}{dv_1{}^2}\frac{d^2\pi}{dt^2} - \frac{d^2\pi}{dt dv_1}\frac{d^2\pi}{dv_1 dt} > 0$$

$$\frac{d^2\pi}{dv_1{}^2}\frac{d^2\pi}{dt^2} > \frac{d^2\pi}{dt dv_1}\frac{d^2\pi}{dv_1 dt} = \left(\frac{d^2\pi}{dv_1 dt}\right)^2$$

Comparative statics analysis based on two dimensional optimization:

$$\begin{bmatrix} \dfrac{d^2\pi}{dv_1^{\,2}} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dtdv_1} & \dfrac{d^2\pi}{dt^2} \end{bmatrix} \begin{bmatrix} dv_1^{*} \\ dt^{*} \end{bmatrix} = \begin{bmatrix} -\dfrac{d^2\pi}{dv_1 dc}dc - \dfrac{d^2\pi}{dv_1 dp}dp - \dfrac{d^2\pi}{dv_1 dr}dr \\ -\dfrac{d^2\pi}{dtdc}dc - \dfrac{d^2\pi}{dtdp}dp - \dfrac{d^2\pi}{dtdr}dr \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{d^2\pi}{dv_1{}^2} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dtdv_1} & \dfrac{d^2\pi}{dt^2} \end{bmatrix} \begin{bmatrix} dv_1{}^* \\ dt^* \end{bmatrix} = \begin{bmatrix} -\dfrac{d^2\pi}{dv_1 dc} dc \\ -\dfrac{d^2\pi}{dtdc} dc \end{bmatrix}$$

$$\frac{dv_1^{\ *}}{dc} = \frac{\begin{vmatrix} 0 & \dfrac{d^2\pi}{dv_1 dt} \\ -\dfrac{d^2\pi}{dtdc} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{|D|} = \frac{\dfrac{d^2\pi}{dtdc}\dfrac{d^2\pi}{dv_1 dt}}{|D|} < 0$$

$$\frac{dt^*}{dc} = \frac{\begin{vmatrix} \dfrac{d^2\pi}{dv_1^{\,2}} & -\dfrac{d^2\pi}{dv_1 dc} \\ \dfrac{d^2\pi}{dtdv_1} & -\dfrac{d^2\pi}{dtdc} \end{vmatrix}}{\begin{vmatrix} \dfrac{d^2\pi}{dv_1^{\,2}} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dtdv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}} = \frac{\begin{vmatrix} \dfrac{d^2\pi}{dv_1^{\,2}} & 0 \\ \dfrac{d^2\pi}{dtdv_1} & -\dfrac{d^2\pi}{dtdc} \end{vmatrix}}{|D|} = \frac{-\dfrac{d^2\pi}{dv_1^{\,2}}\dfrac{d^2\pi}{dtdc}}{|D|} > 0$$

$$\frac{dv_1^{\ *}}{dp} = \frac{\begin{vmatrix} -\dfrac{d^2\pi}{dv_1 dp} & \dfrac{d^2\pi}{dv_1 dt} \\ -\dfrac{d^2\pi}{dt dp} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{|D|} = \frac{-\dfrac{d^2\pi}{dv_1 dp}\dfrac{d^2\pi}{dt^2} + \dfrac{d^2\pi}{dt dp}\dfrac{d^2\pi}{dv_1 dt}}{|D|} > 0$$

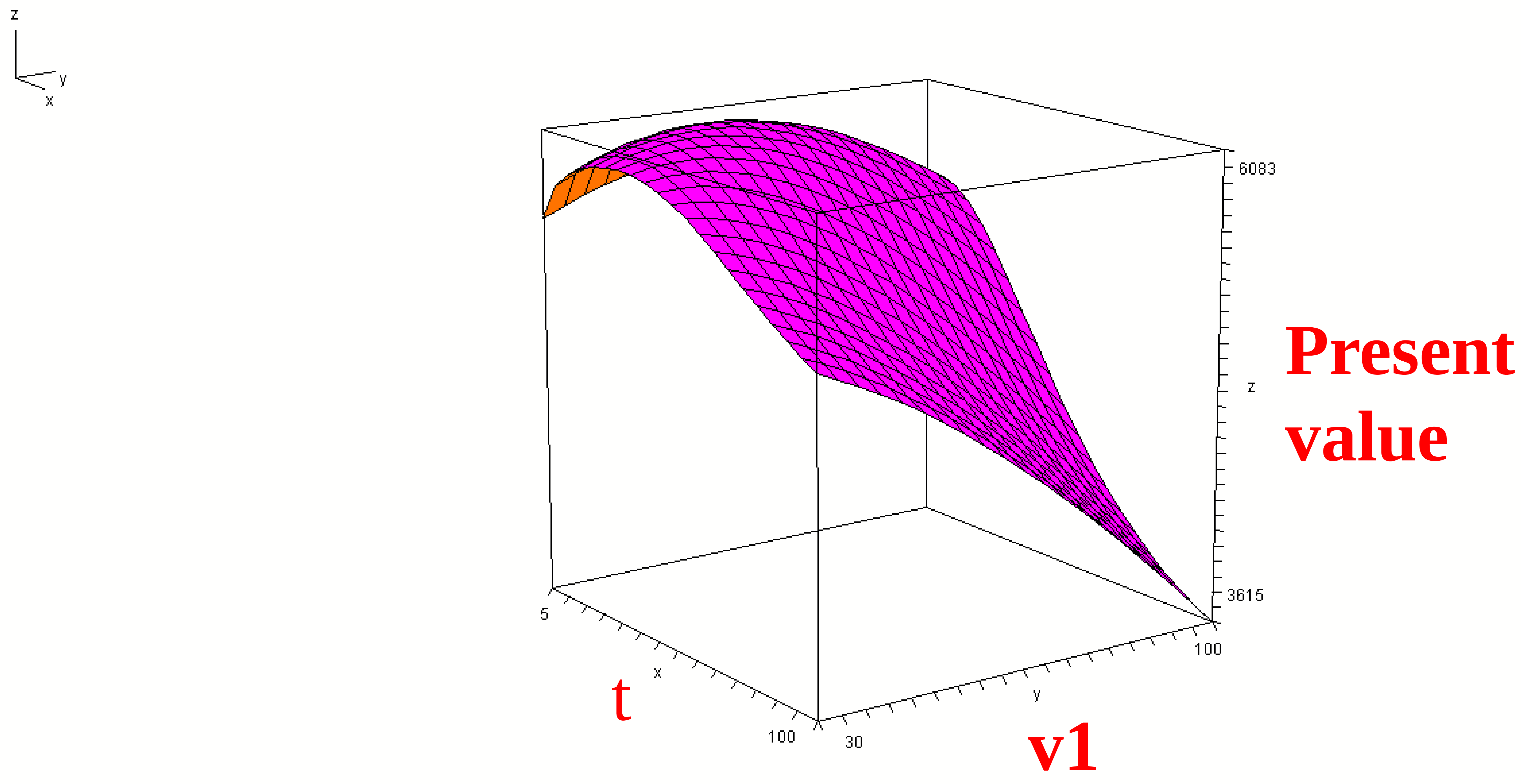
$$\frac{dt^*}{dp} = \frac{\begin{vmatrix} \dfrac{d^2\pi}{dv_1{}^2} & -\dfrac{d^2\pi}{dv_1dp} \\ \dfrac{d^2\pi}{dtdv_1} & -\dfrac{d^2\pi}{dtdp} \end{vmatrix}}{|D|} = \frac{-\dfrac{d^2\pi}{dv_1{}^2}\dfrac{d^2\pi}{dtdp} + \dfrac{d^2\pi}{dtdv_1}\dfrac{d^2\pi}{dv_1dp}}{|D|} < 0$$

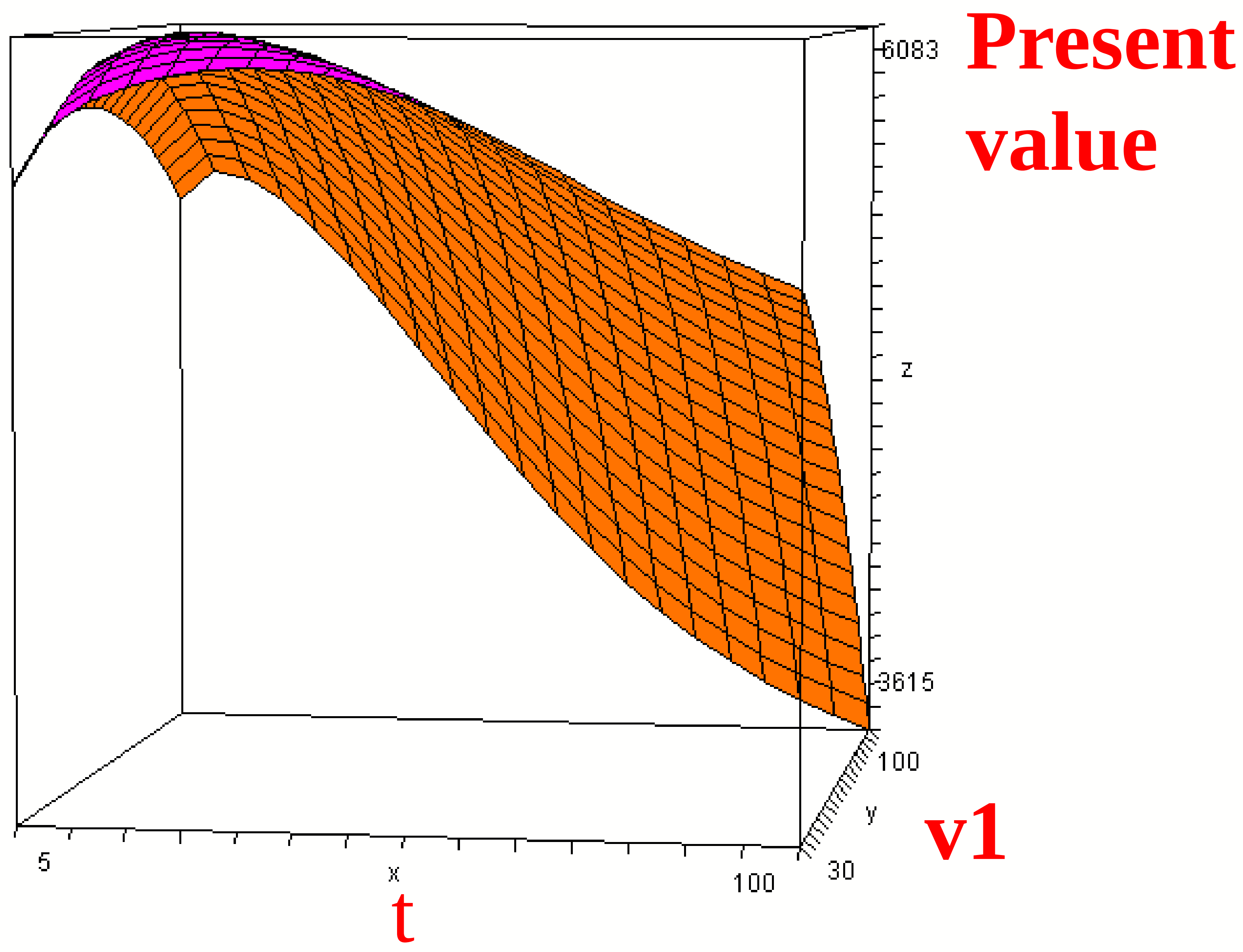
$$\frac{dv_1^*}{dr} = \frac{\begin{vmatrix} -\dfrac{d^2\pi}{dv_1 dr} & \dfrac{d^2\pi}{dv_1 dt} \\ -\dfrac{d^2\pi}{dt dr} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}}{|D|} = \frac{-\dfrac{d^2\pi}{dv_1 dr}\dfrac{d^2\pi}{dt^2} + \dfrac{d^2\pi}{dt dr}\dfrac{d^2\pi}{dv_1 dt}}{|D|} < 0$$

$$\frac{dt^*}{dr} = \frac{\begin{vmatrix} \dfrac{d^2\pi}{dv_1{}^2} & -\dfrac{d^2\pi}{dv_1 dr} \\ \dfrac{d^2\pi}{dtdv_1} & -\dfrac{d^2\pi}{dtdr} \end{vmatrix}}{\begin{vmatrix} \dfrac{d^2\pi}{dv_1{}^2} & \dfrac{d^2\pi}{dv_1 dt} \\ \dfrac{d^2\pi}{dtdv_1} & \dfrac{d^2\pi}{dt^2} \end{vmatrix}} = \frac{-\dfrac{d^2\pi}{dv_1{}^2}\dfrac{d^2\pi}{dtdr} + \dfrac{d^2\pi}{dtdv_1}\dfrac{d^2\pi}{dv_1 dr}}{\left|D\right|} < 0$$

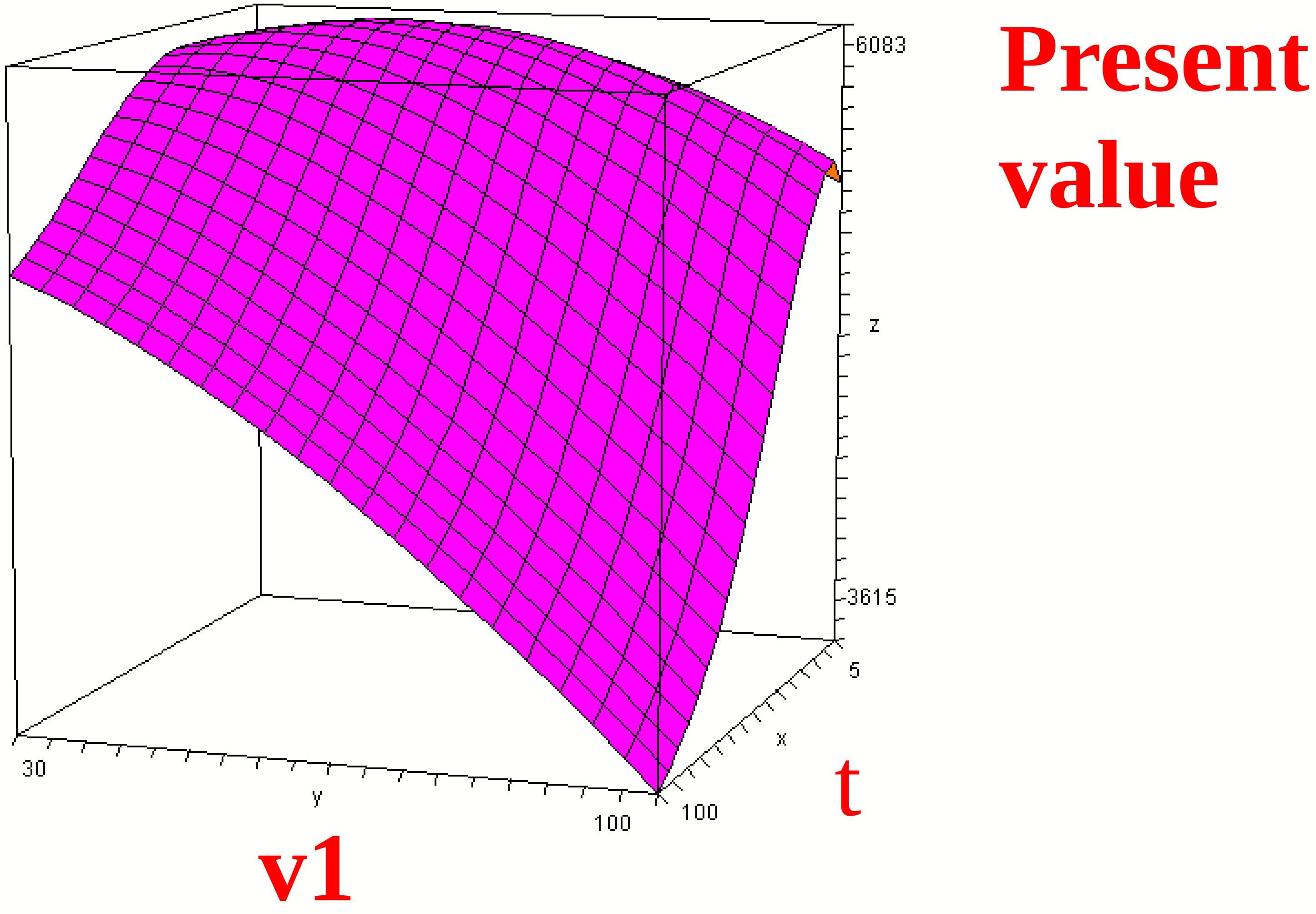
Graphical illustrations based on specified functions and parameters

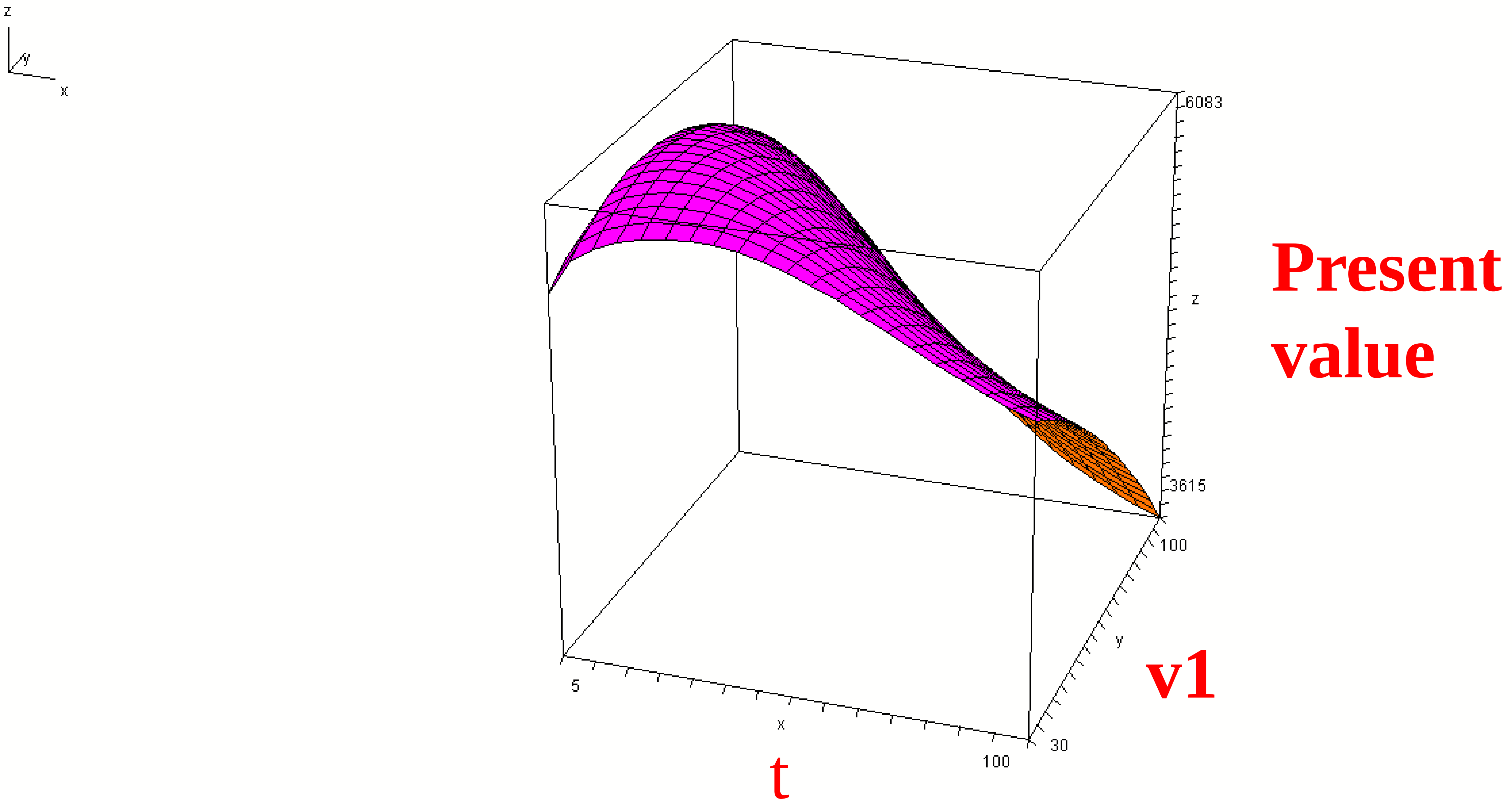
$$\left(30\cdot(200-v)+\frac{0.1833333}{2}\cdot(200-v)\cdot(200-v)-\frac{0.001666667}{3}\cdot(200-v)\cdot(200-v)\cdot(200-v)-50\right)+\frac{20\cdot\left(\dfrac{1}{\dfrac{1}{400}+\left(\dfrac{1}{v}-\dfrac{1}{400}\right)\cdot EXP(-0.05\cdot t)}-v\right)-50}{EXP(0.03\cdot t)-1}$$


z
y
x
6083
z
3615
5
x
100
30
y
100
Present value
t
v1



Present value
6083
z
3615
100
y
v1
30
5
x
100
t


6083
z
-3615
5
x
100
30
y
100
Present value
t
v1


z
y
x
6083
z
3615
100
y
30
5
x
100
Present value
v1
t

Numerical Analysis

Peter Lohmander 150812

Case 0:

```
! OPT CCF 150812;
! Peter Lohmander;

v0 = 200;
p = 20;
c = 50;
r = 0.03;
m0 = 30;
c0 = 50;

max = Y;
Y = R0 + (p*Q-c)/(@exp(r*t)-1);
h = v0-v1;
h < v0;
h > 1;

mp = m0 - a*h-b*h*h;
R0 = m0*h-a/2*h*h-b/3*h*h*h-c0;
Q = 1/(1/400+(1/v1-1/400)*@exp(-0.05*t))-v1;

! Derivation of initial marginal price function;
150*a+(150)^2*b= 10;
200*a+(200)^2*b = 30;
@free(a);
@free(b);
```

```
Local optimal solution found.
  Objective value:                              6084.286
  Infeasibilities:                              0.000000
  Total solver iterations:                            34
```

Variable	Value	Reduced Cost
V0	200.0000	0.000000
P	20.00000	0.000000
C	50.00000	0.000000
R	0.3000000E-01	0.000000
M0	30.00000	0.000000
C0	50.00000	0.000000
Y	6084.286	0.000000
R0	4651.524	0.000000
Q	71.17267	0.000000
T	22.40775	0.1339671E-08
H	150.7051	0.000000
V1	49.29487	0.000000
MP	19.77588	0.000000
A	-0.1833333	0.000000
B	0.1666667E-02	0.000000

Case c = 25:

```
Local optimal solution found.
Objective value:                              6112.837
Infeasibilities:                              0.1421085E-13
Total solver iterations:                            33
```

Variable	Value	Reduced Cost
V0	200.0000	0.000000
P	20.00000	0.000000
C	25.00000	0.000000
R	0.3000000E-01	0.000000
M0	30.00000	0.000000
C0	50.00000	0.000000
Y	6112.837	0.000000
R0	4614.702	0.000000
Q	60.39070	0.000000
T	19.39833	0.000000
H	148.8702	0.000000
V1	51.12982	0.000000
MP	20.35565	0.000000
A	-0.1833333	0.000000
B	0.1666667E-02	0.000000

Case c = 75:

```
Local optimal solution found.
Objective value:                                    6059.922
Infeasibilities:                                0.1818989E-11
Total solver iterations:                              36
```

Variable	Value	Reduced Cost
V0	200.0000	0.000000
P	20.00000	0.000000
C	75.00000	0.000000
R	0.3000000E-01	0.000000
M0	30.00000	0.000000
C0	50.00000	0.000000
Y	6059.922	0.000000
R0	4678.431	0.000000
Q	79.22597	0.000000
T	24.61475	0.000000
H	152.0811	0.000000
V1	47.91893	0.000000
MP	19.33378	0.000000
A	-0.1833333	0.000000
B	0.1666667E-02	0.000000

Case r = 0.01:

```
Local optimal solution found.
Objective value:                                11288.77
Infeasibilities:                              0.1818989E-11
Total solver iterations:                              39
```

Variable	Value	Reduced Cost
V0	200.0000	0.000000
P	20.00000	0.000000
C	50.00000	0.000000
R	0.1000000E-01	0.000000
M0	30.00000	0.000000
C0	50.00000	0.000000
Y	11288.77	0.000000
R0	3708.662	0.000000
Q	122.3903	0.000000
T	27.48465	0.000000
H	112.9925	0.000000
V1	87.00750	0.000000
MP	29.43645	0.000000
A	-0.1833333	0.000000
B	0.1666667E-02	0.000000

Case r = 0.05:

```
Local optimal solution found.
Objective value:                                    5438.774
Infeasibilities:                                 0.1421085E-13
Total solver iterations:                                  31
```

Variable	Value	Reduced Cost
V0	200.0000	0.000000
P	20.00000	0.000000
C	50.00000	0.000000
R	0.5000000E-01	0.000000
M0	30.00000	0.000000
C0	50.00000	0.000000
Y	5438.774	0.000000
R0	4935.135	0.000000
Q	30.30823	-0.2519769E-08
T	14.87958	0.1972276E-07
H	167.4356	0.000000
V1	32.56436	0.000000
MP	13.97205	0.000000
A	-0.1833333	0.000000
B	0.1666667E-02	0.000000

Case p= 15:

```
Local optimal solution found.
Objective value:                              5744.222
Infeasibilities:                             0.1421085E-13
Total solver iterations:                            33
```

Variable	Value	Reduced Cost
V0	200.0000	0.000000
P	15.00000	0.000000
C	50.00000	0.000000
R	0.3000000E-01	0.000000
M0	30.00000	0.000000
C0	50.00000	0.000000
Y	5744.222	0.000000
R0	4855.214	0.000000
Q	87.63010	0.000000
T	29.49081	0.2944944E-08
H	162.0924	0.000000
V1	37.90758	0.000000
MP	15.92702	0.000000
A	-0.1833333	0.000000
B	0.1666667E-02	0.000000

Case p = 25:

```
Local optimal solution found.
Objective value:                              6485.002
Infeasibilities:                              0.9094947E-12
Total solver iterations:                            33
```

Variable	Value	Reduced Cost
V0	200.0000	0.000000
P	25.00000	0.000000
C	50.00000	0.000000
R	0.3000000E-01	0.000000
M0	30.00000	0.000000
C0	50.00000	0.000000
Y	6485.002	0.000000
R0	4412.308	0.000000
Q	58.10996	0.000000
T	17.22910	0.000000
H	139.5701	0.000000
V1	60.42991	0.2916067E-08
MP	23.12150	0.000000
A	-0.1833333	0.000000
B	0.1666667E-02	0.000000

Case m0 = 25:

```
Local optimal solution found.
Objective value:                                5358.305
Infeasibilities:                               0.9094947E-12
Total solver iterations:                              39
```

Variable	Value	Reduced Cost
V0	200.0000	0.000000
P	20.00000	0.000000
C	50.00000	0.000000
R	0.3000000E-01	0.000000
M0	25.00000	0.000000
C0	50.00000	0.000000
Y	5358.305	0.000000
R0	3713.153	0.000000
Q	61.67188	0.000000
T	18.06485	0.000000
H	139.4981	0.000000
V1	60.50189	0.000000
MP	18.14178	0.000000
A	-0.1833333	0.000000
B	0.1666667E-02	0.000000

Case m0 = 35:

```
Local optimal solution found.
Objective value:                                6863.473
Infeasibilities:                               0.9094947E-12
Total solver iterations:                              35
```

Variable	Value	Reduced Cost
V0	200.0000	0.000000
P	20.00000	0.000000
C	50.00000	0.000000
R	0.3000000E-01	0.000000
M0	35.00000	0.000000
C0	50.00000	0.000000
Y	6863.473	0.000000
R0	5637.867	0.000000
Q	82.08137	0.000000
T	27.74400	0.000000
H	160.7782	0.000000
V1	39.22175	0.000000
MP	21.39327	0.000000
A	-0.1833333	0.000000
B	0.1666667E-02	0.000000

The following pages include general second order conditions with notation following Chiang, A., Fundamental methods of mathematical economics, McGraw-Hill, 2 ed., 1974

Appendix:

On the second order total differential and the second order maximum and minimum conditions:

$$d^2z = f_{xx}dx^2 + f_{xy}dxdy + f_{yx}dydx + f_{yy}dy^2$$

$$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$$

$$q = au^2 + 2huv + bv^2$$

$$q = au^2 + 2huv + bv^2$$

$$q = au^2 + 2huv + \frac{h^2}{a}v^2 + bv^2 - \frac{h^2}{a}v^2$$

$$q = a\left(u^2 + \frac{2h}{a}uv + \frac{h^2}{a^2}v^2\right) + \left(b - \frac{h^2}{a}\right)v^2$$

$$q = a\left(u + \frac{h}{a}v\right)^2 + \left(\frac{ab - h^2}{a}\right)v^2$$

Results:

If: $\left(\left(a<0\right)\wedge\left(ab-h^{2}>0\right)\right)\Rightarrow\left(q<0\right)$, the quadratic form is said to be negative definite.

If: $\left(\left(a>0\right)\wedge\left(ab-h^{2}>0\right)\right)\Rightarrow\left(q>0\right)$, the quadratic form is said to be positive definite.

In other words: Assumption: A unique (local) maximum exists. The following conditions hold:

$$\left|\frac{d^2 f}{dx^2}\right|<0,\quad \left|\frac{d^2 f}{dy^2}\right|<0,\quad \begin{vmatrix} \dfrac{d^2 f}{dx^2} & \dfrac{d^2 f}{dxdy} \\ \dfrac{d^2 f}{dydx} & \dfrac{d^2 f}{dy^2} \end{vmatrix}>0$$

For a unique (local) maximum, it is sufficient that:

$$\left|\frac{d^2 f}{dx^2}\right| < 0, \qquad \begin{vmatrix} \dfrac{d^2 f}{dx^2} & \dfrac{d^2 f}{dxdy} \\ \dfrac{d^2 f}{dydx} & \dfrac{d^2 f}{dy^2} \end{vmatrix} > 0$$

For a unique (local) minimum, it is sufficient that:

$$\left|\frac{d^2 f}{dx^2}\right| > 0, \qquad \begin{vmatrix} \dfrac{d^2 f}{dx^2} & \dfrac{d^2 f}{dxdy} \\ \dfrac{d^2 f}{dydx} & \dfrac{d^2 f}{dy^2} \end{vmatrix} > 0$$

MATHEMATICAL APPENDIX

to

Optimal continuous cover forest management:
- Economic and environmental effects and legal considerations

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BIT's 5th Low Carbon Earth Summit

(LCES 2015 & ICE-2015)

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