Linear and nonlinear optimization of international trade and tariffs

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- Invited Talk

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Lohmander, P., Linear and nonlinear optimization of international trade and tariffs, WSTA 2025

ABSTRACT

International trade, with and without tariffs, is analytically and numerically defined and analyzed. First, the linear partial equilibrium two nation trade model approach by Pindyck and Rubinfeld and by Krugman and Obstfeld, is applied. The producer and consumer surpluses are determined in the two nations, N1 and N2, with and without trade. With free trade without tariffs, the total surpluses in the two nations, including producer and consumer surpluses, exceed the total surpluses in the two countries, without trade. Then, N1 introduces tariffs on imports. The objective function of N1 is the total surplus of N1, including consumer and producer surplus and the tariff gain. The optimal tariff for N1 is determined as a general function of all parameters of demand and supply in the two nations. The objective function of N1 is a strictly concave and quadratic function of the tariff level, which implies that the optimal tariff is unique and instantly determined from the first order optimum condition. In N1, it is proved that the optimal tariff is strictly positive. In N1, with the optimal tariff, the tariff gain and the producer surplus increase and the consumer surplus decreases, compared to the free trade solution. In N2, the consumer surplus increases and the producer surplus decreases, if the tariff in N1 increases. The total surplus in N2 however decreases. With a strictly positive N1 tariff, the total surplus in both nations, is reduced, compared to the free trade solution. Then, a nonlinear general equilibrium trade model with three nations and four products is defined and analyzed. Each nation, N1, N2 and N3, can produce two different products. The country specific production possibility frontiers are strictly concave and guarantee strictly positive production levels for all feasible relative prices. The demand functions in the different countries are determined from maximized nonlinear Cobb Douglas utility functions with country specific parameters and the national consumption budgets. These budgets are functions of all relative prices, zero excess supplies in all international product markets, and optimized production levels in all countries. Three equilibrium relative prices are numerically determined from the total nonlinear system. These prices guarantee that the total excess supply functions approach zero in all international product markets. Rapid and reliable relative price convergence is obtained via new and generalized multivariate bisection method in three dimensions. In this process, the objective function is defined as the sum of squares of all excess supply function values. The free trade equilibrium and the equilibrium affected by tariffs introduced by N1 are determined. The optimal tariff from the N1 perspective is determined. In free trade, N3 exports product 4 to N1 but N1 does not export anything to N3. Then, N1 optimizes a tariff on product 4, only imported from N3. The optimal tariff from the N1 perspective is strictly positive and the utility of N1 increases compared to the free trade solution. The total utility of all consumers in all countries decreases when the tariff increases. The optimal tariff determined by N1 is much larger than in the linear model, and is a function of all parameters in all countries. Several other tariff effects on all countries and products, including production and consumption changes in N2, are reported.

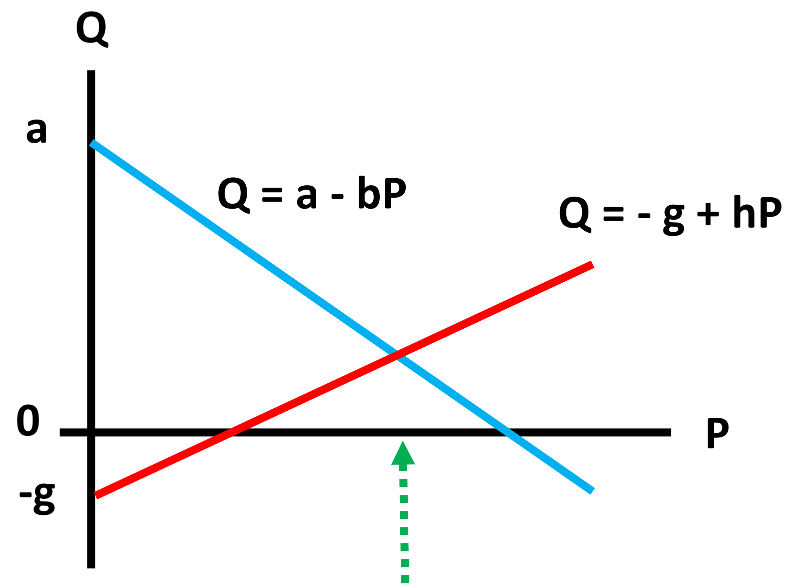
Fundamental derivations and proofs based on partial equilibrium analysis:

- Why is it optimal, for at least one nation, to introduce tariffs, if other nations do not use tariffs?
- Why is it optimal, for at least one nation, to reduce the import volume compared to the free trade solution?
- What is: The total surplus as a function of the import volume.
- What is: The optimal improvement of the sum of the consumer surplus and the tariff revenue.
- What is: The optimal tariff level.
- What is: The optimal price in the world market
- What is: The optimal price in the home country.
- What is: The optimal import volume.

Linear approximations of demand and supply

$$Q = a - bP$$ **Demand function in N1**

$$Q = -g + hP$$ **Export supply from N2 (or from the ”World”)**



$$-g + hP = a - bP$$ **Market equilibrium condition**

$$(b + h)P = a + g$$

$$P = \frac{a + g}{b + h}$$ **Equilibrium price**

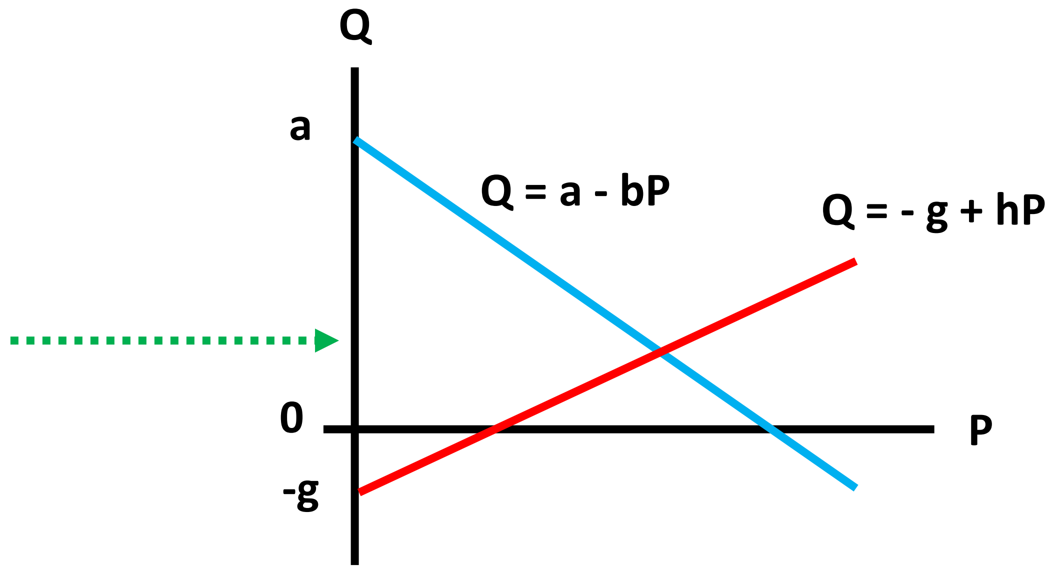
$$P^F = \frac{a+g}{b+h}$$ **Free trade equilibrium price**

$$Q_D^F = a - b\left(\frac{a+g}{b+h}\right)$$ **Demand in N1 in free trade equilibrium**

$$Q_D^F = \frac{a(b+h) - b(a+g)}{b+h}$$

$$\boxed{Q_D^F = \frac{ah - bg}{b+h}}$$

Demand in N1 in free trade equilibrium



$$P^F = \frac{a+g}{b+h}$$ **Free trade equilibrium price**

$$Q_S^F = -g + h\left(\frac{a+g}{b+h}\right)$$ **Export supply from N2 (or ”the from World”) in free trade equilibrium**

$$Q_S^F = \frac{-g(b+h)+h(a+g)}{b+h}$$

$$Q_S^F = \frac{ah-bg}{b+h} \quad \left(= Q_D^F\right)$$ **Export supply from N2 (World) in free trade equilibrium. This equals the demand in N1.**

$$Q = a - bP$$

Demand function in N1

$$bP = a - Q$$

$$P = \frac{a}{b} - \frac{1}{b}Q$$

Inverse Demand function in N1

$$Q = -g + hP$$

Supply function from N2

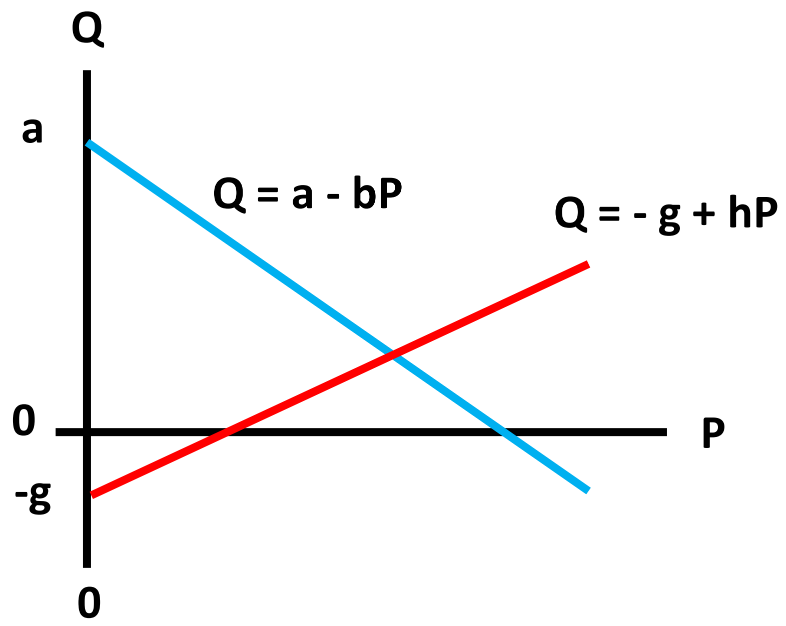
$$hP = g + Q$$

$$P = \frac{g}{h} + \frac{1}{h}Q$$

Inverse Supply function from N2

$$Q = a - bP$$ **Demand**

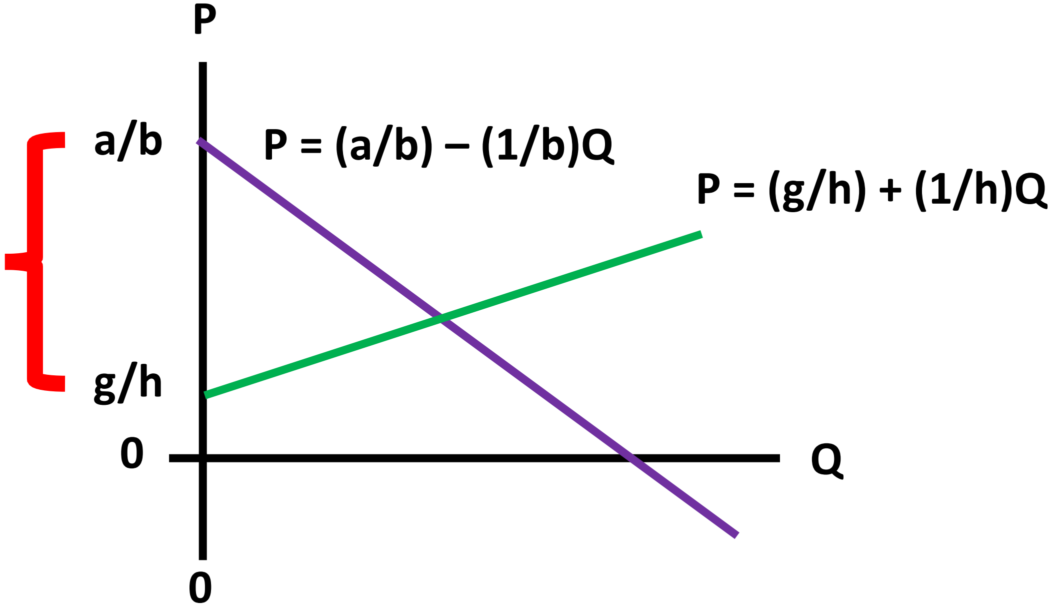
$$P = \frac{a}{b} - \frac{1}{b}Q$$ **Inverse Demand**



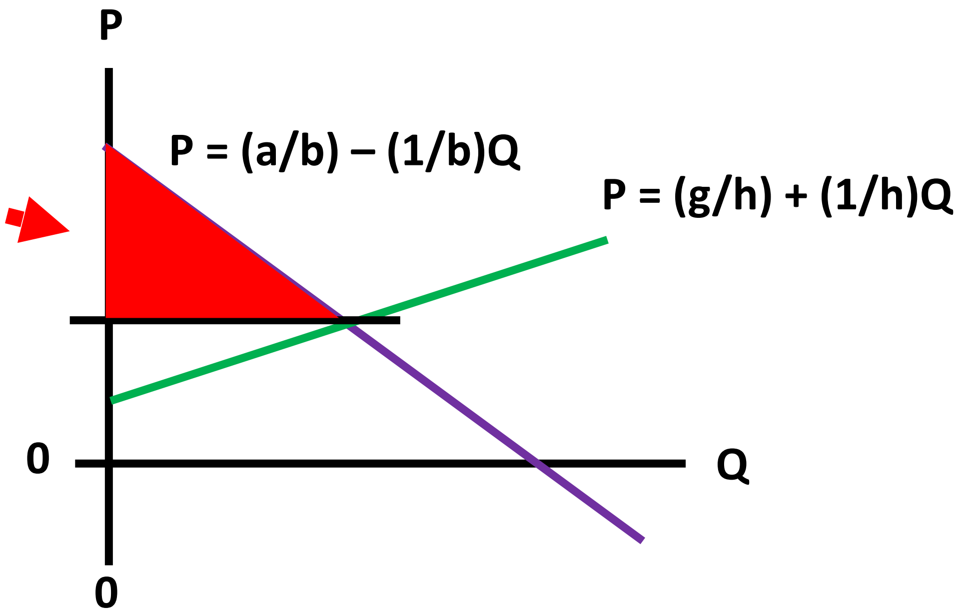
$$Q = -g + hP$$ **Supply**

$$P = \frac{g}{h} + \frac{1}{h}Q$$ **Inverse Supply**

$$\frac{a}{b} > \frac{g}{h}$$



Consumer surplus in N1 in free trade equilibrium



$$S = \int_0^Q \left(\frac{a}{b} - \frac{1}{b} q \right) dq - \left(\frac{g}{h} + \frac{1}{h} Q \right) Q \quad , Q = Q_D^F = \frac{ah - bg}{b + h}$$

$$S = \int_0^Q \left(\frac{a}{b} - \frac{1}{b} q \right) dq - \left(\frac{g}{h} + \frac{1}{h} Q \right) Q$$

$$S = \left[\frac{a}{b} q - \frac{1}{2b} q^2 \right]_0^Q - \frac{g}{h} Q - \frac{1}{h} Q^2$$

$$S = \frac{a}{b} Q - \frac{1}{2b} Q^2 - \frac{g}{h} Q - \frac{1}{h} Q^2$$

$$S = \left(\frac{a}{b} - \frac{g}{h} \right) Q - \left(\frac{1}{2b} + \frac{1}{h} \right) Q^2$$

$$S = \left(\frac{a}{b} - \frac{g}{h}\right)Q - \left(\frac{1}{2b} + \frac{1}{h}\right)Q^2$$

Consumer surplus in N1 in free trade equilibrium:

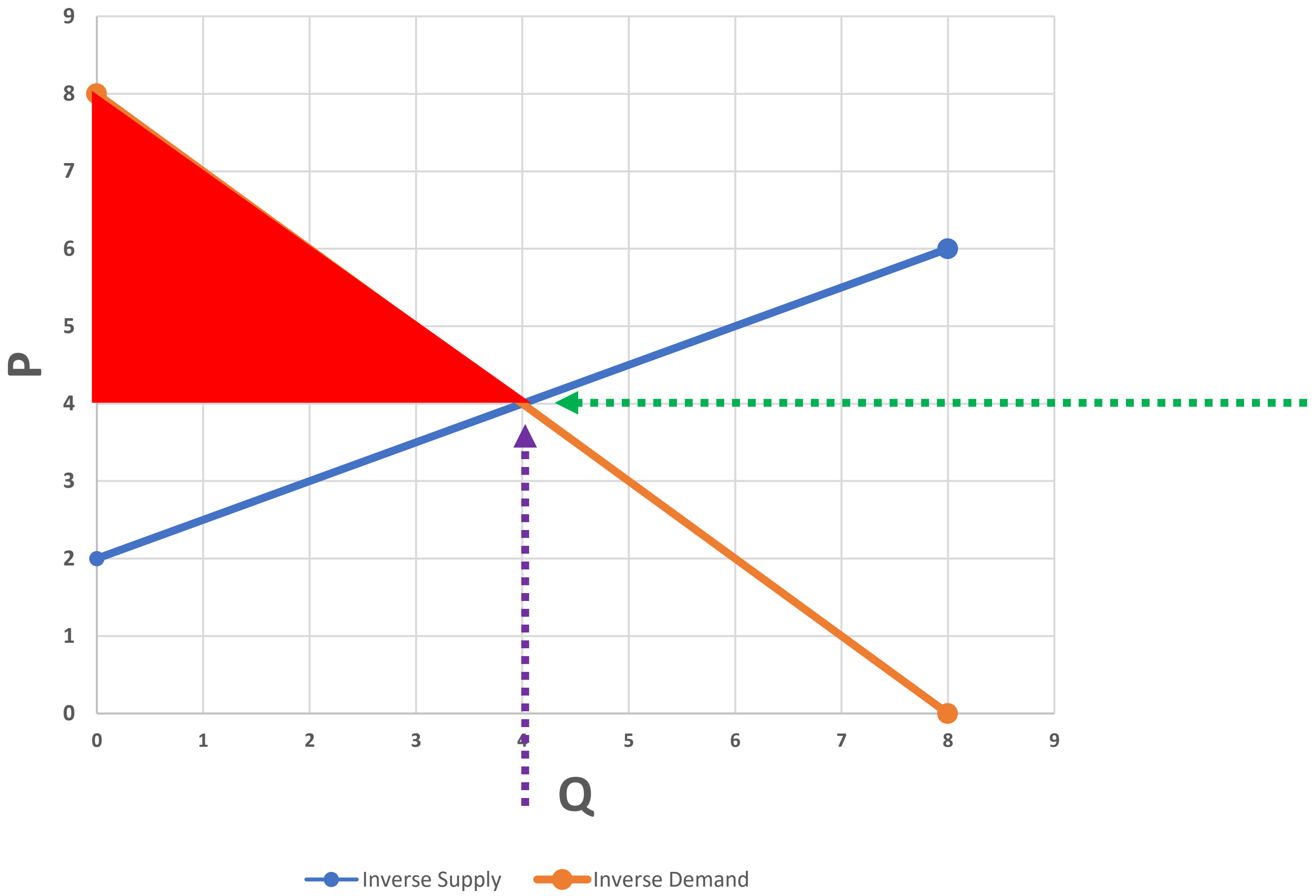
$$S^F = \left(\frac{a}{b} - \frac{g}{h}\right)\left(\frac{ah - bg}{b + h}\right) - \left(\frac{1}{2b} + \frac{1}{h}\right)\left(\frac{ah - bg}{b + h}\right)^2$$

$$Q_D^F = \frac{ah - bg}{b + h}$$

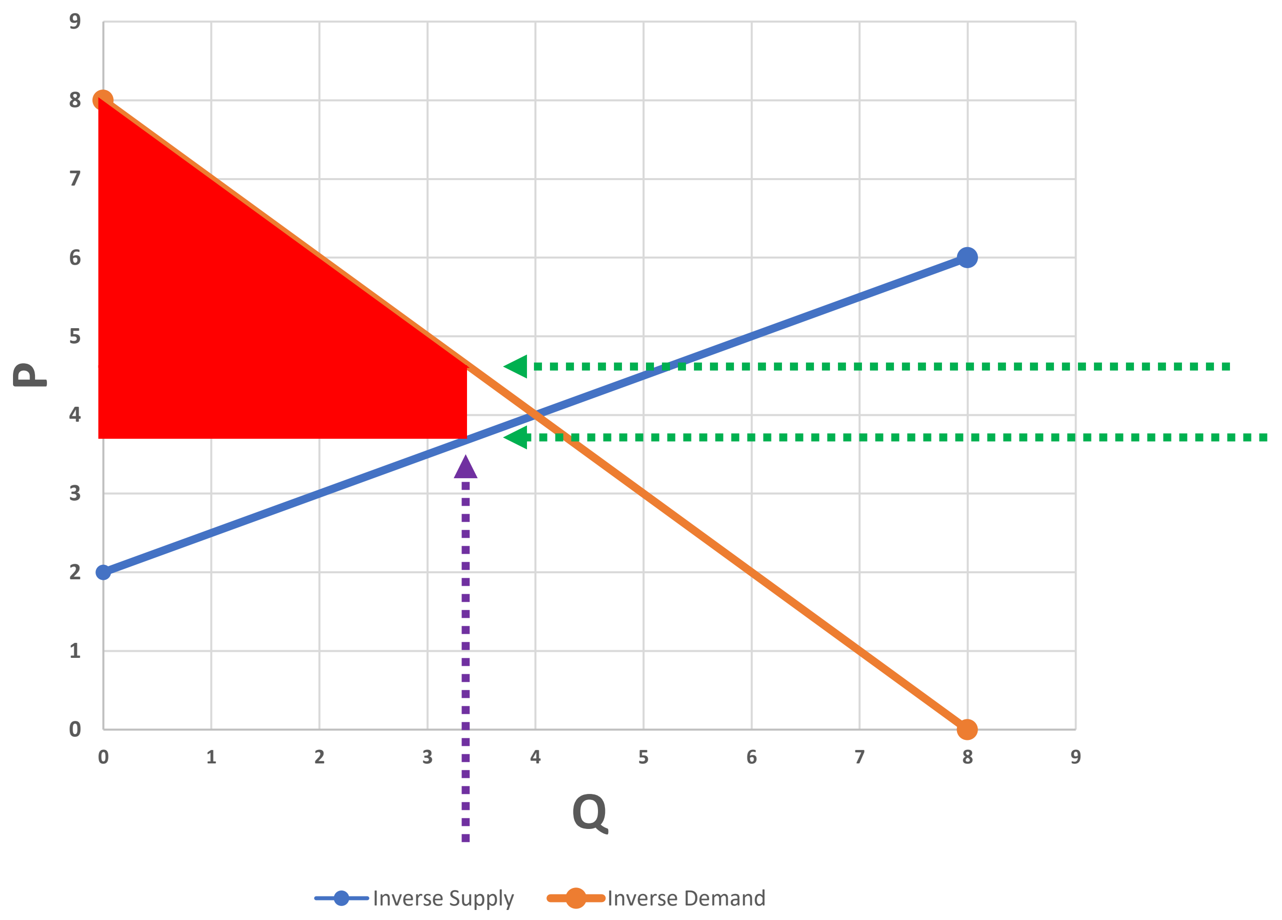
Maybe, it is possible to find an import volume, Q,

that gives a larger consumer surplus

than the free trade equilibrium?

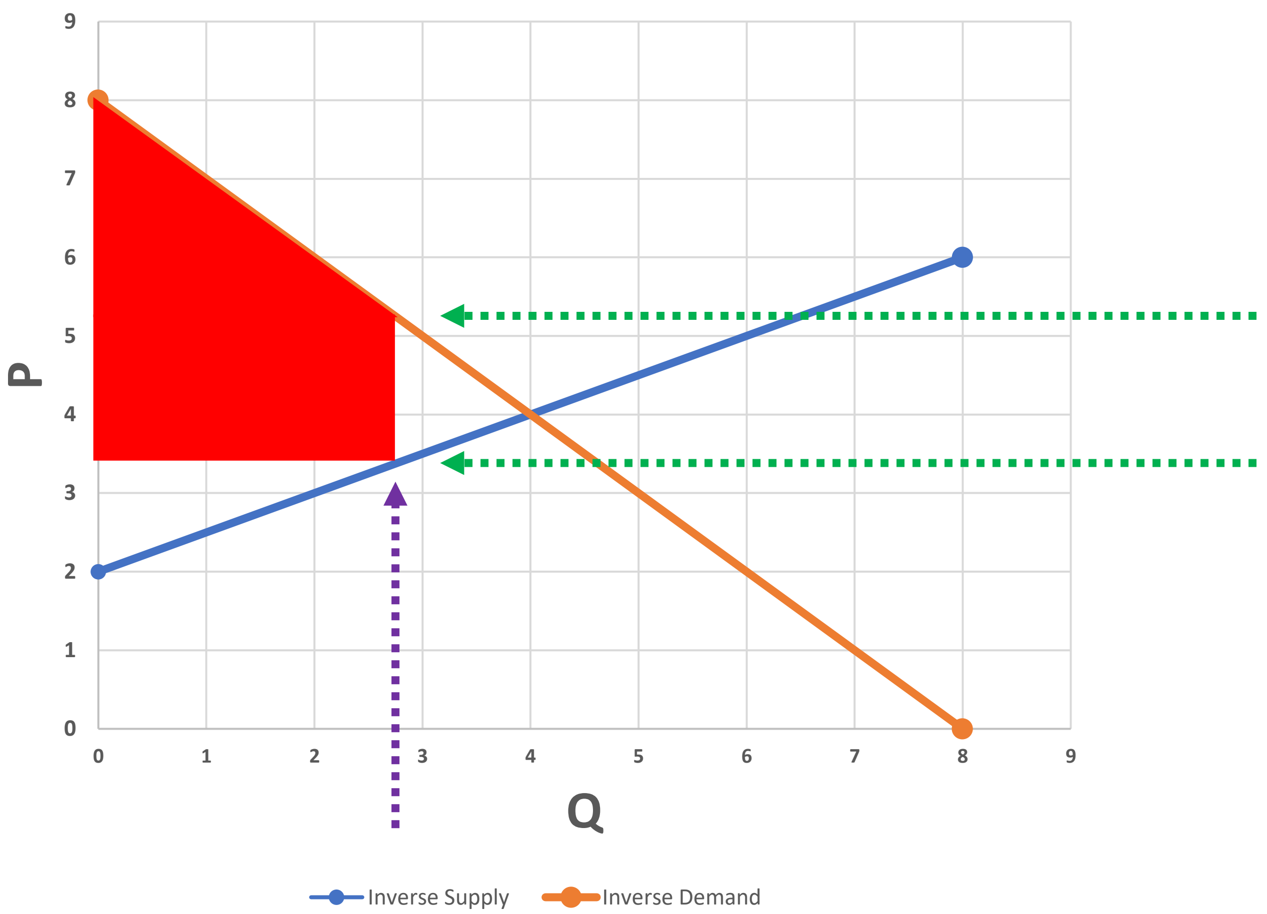


PF, Price ”in Free Trade”



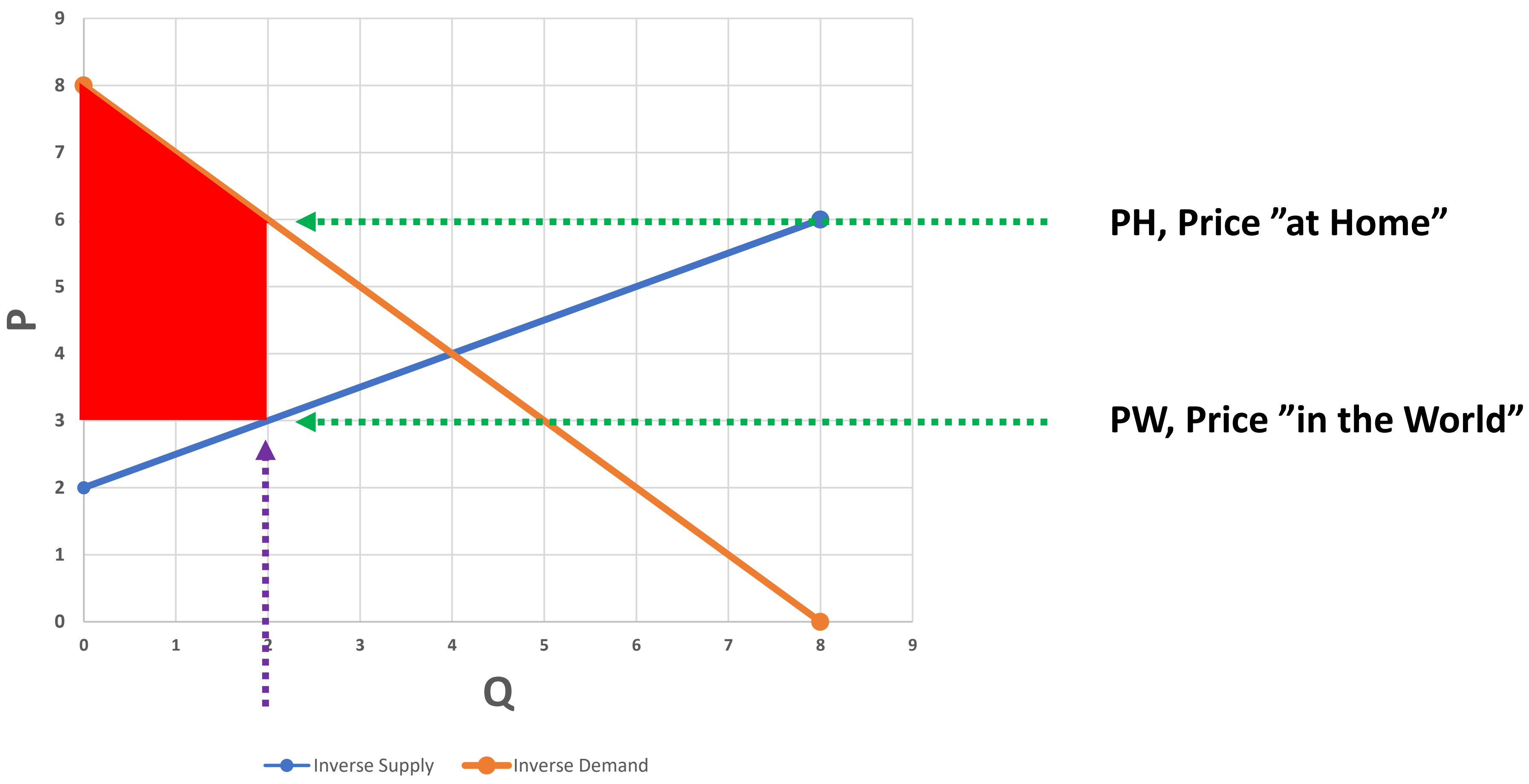
PH, Price ”at Home”

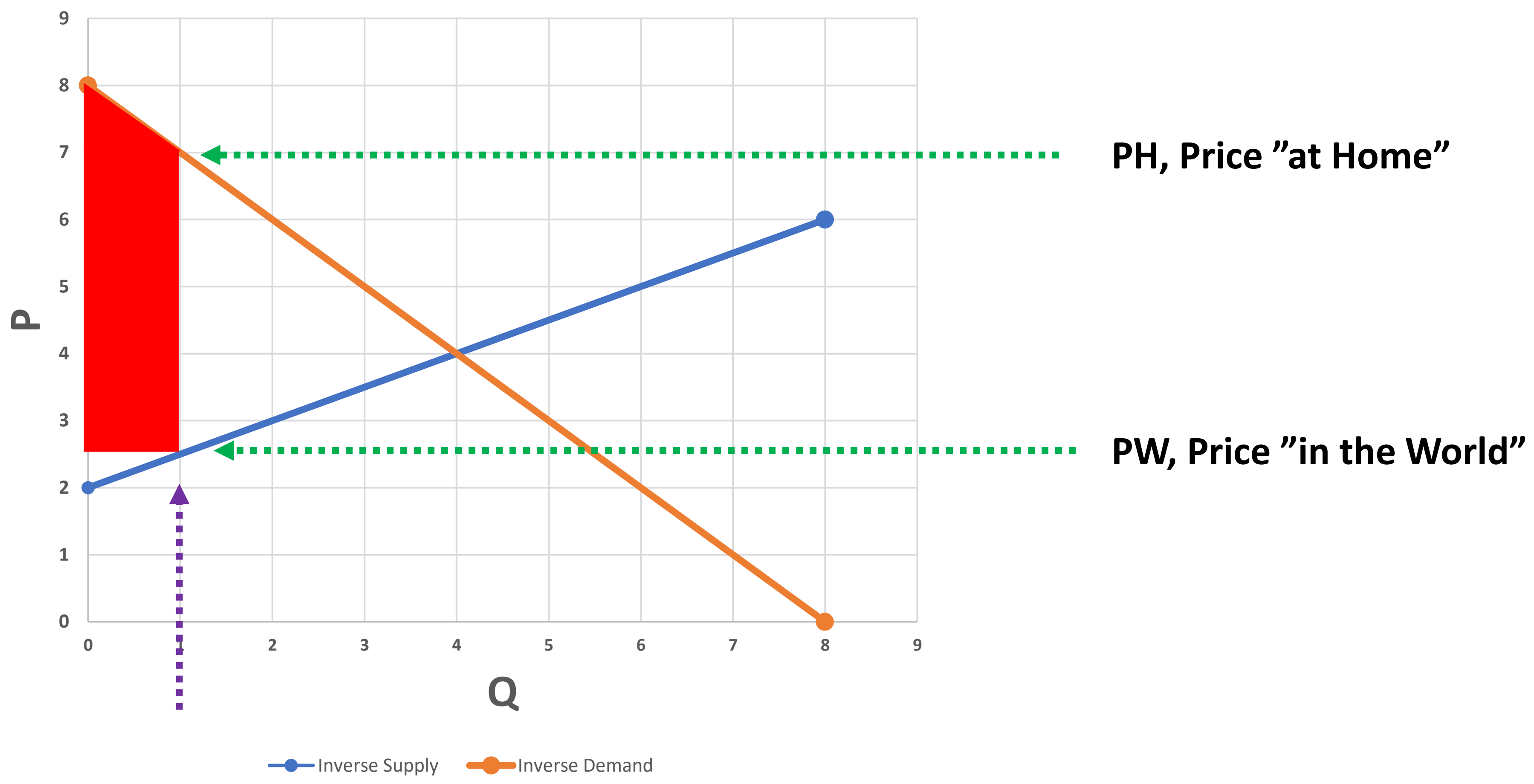
PW, Price ”in the World”



PH, Price ”at Home”

PW, Price ”in the World”


9
8
7
6
5
4
3
2
1
0
P
0
1
2
3
4
5
6
7
8
9
Q
Inverse Supply
Inverse Demand
PH, Price ”at Home”
PW, Price ”in the World”


PH, Price ”at Home”
PW, Price ”in the World”
P
Q
0
1
2
3
4
5
6
7
8
9
Inverse Supply
Inverse Demand

Maximization of consumer surplus in N1, via control of the total import, Q.

$$S = \left(\frac{a}{b} - \frac{g}{h}\right)Q - \left(\frac{1}{2b} + \frac{1}{h}\right)Q^2$$

$$\frac{dS}{dQ} = \left(\frac{a}{b} - \frac{g}{h}\right) - \left(\frac{1}{b} + \frac{2}{h}\right)Q = 0$$

First order optimum condition.

$$\frac{d^2S}{dQ^2} = -\left(\frac{1}{b} + \frac{2}{h}\right) < 0$$

Second order maximum condition.
This is always satisfied,
since b and h are strictly positive.

$$\frac{dS}{dQ} = \left(\frac{a}{b} - \frac{g}{h}\right) - \left(\frac{1}{b} + \frac{2}{h}\right)Q = 0$$

$$\frac{dS}{dQ} = \left(\frac{ah - bg}{bh}\right) - \left(\frac{h + 2b}{bh}\right)Q = 0$$

$$Q = \frac{\left(\frac{ah - bg}{bh}\right)}{\left(\frac{h + 2b}{bh}\right)} = \frac{ah - bg}{2b + h} = Q^*$$

= *Optimal import level.*

$$Q^* = \frac{ah - bg}{2b + h}$$ = *Optimal import level.*

$$Q^F = \frac{ah - bg}{h + b}$$ = *Free trade import level.*

$$0 < \frac{Q^*}{Q^F} = \frac{\left(\frac{ah - bg}{2b + h}\right)}{\left(\frac{ah - bg}{b + h}\right)} = \frac{b + h}{2b + h} < 1$$

Important Observation:

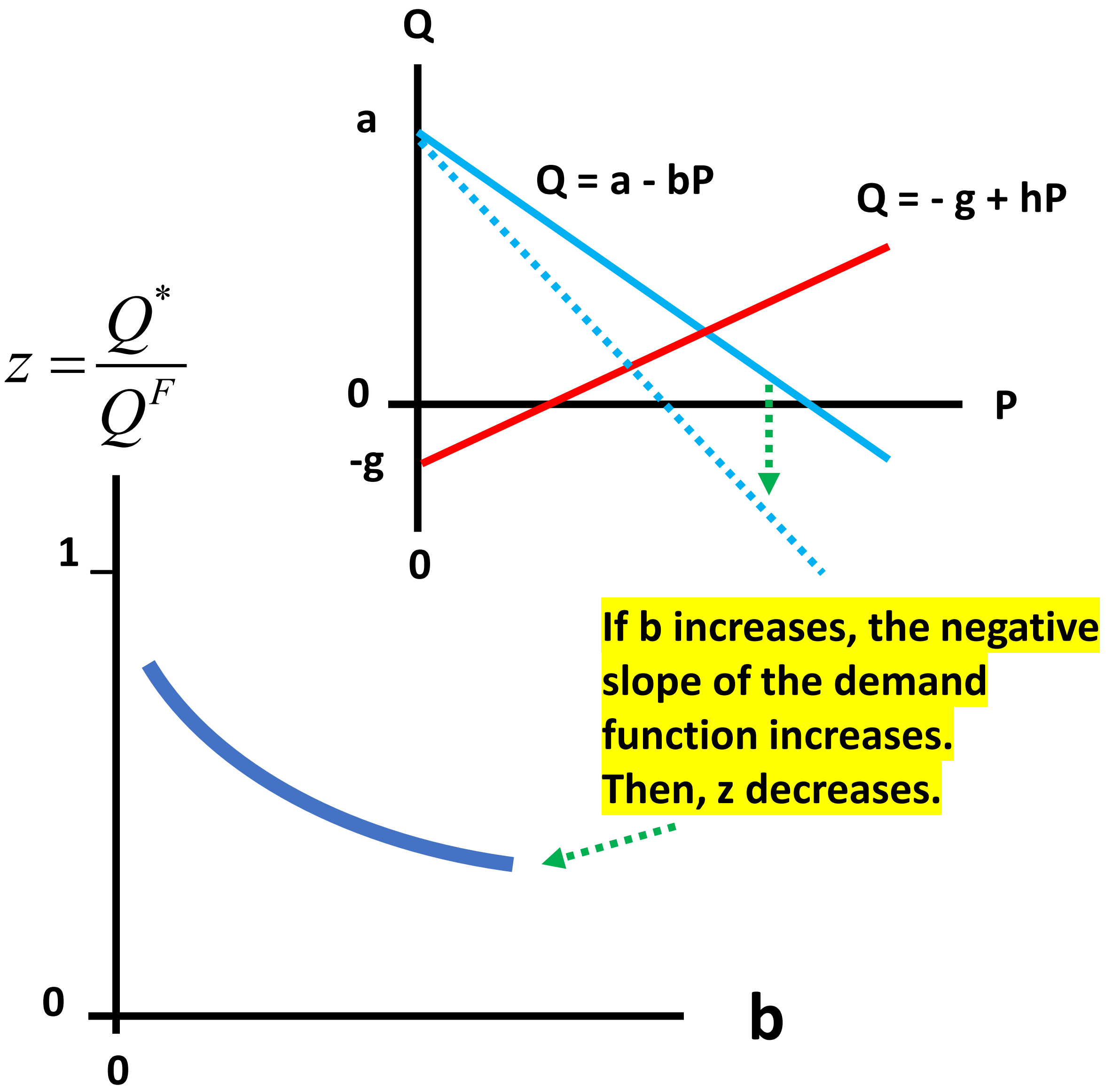
It is always optimal to import less, than according to the free trade solution, as long as the export supply function from other nations is not changed.

$$0 < z = \frac{Q^*}{Q^F} = \frac{b+h}{2b+h} < 1$$

$$\frac{dz}{db} = \frac{(1)(2b+h) - (b+h)(2)}{\left(2b+h\right)^2}$$

$$\frac{dz}{db} = \frac{2b+h-2b-2h}{\left(2b+h\right)^2}$$

$$\frac{dz}{db} = \frac{-h}{\left(2b+h\right)^2} < 0$$

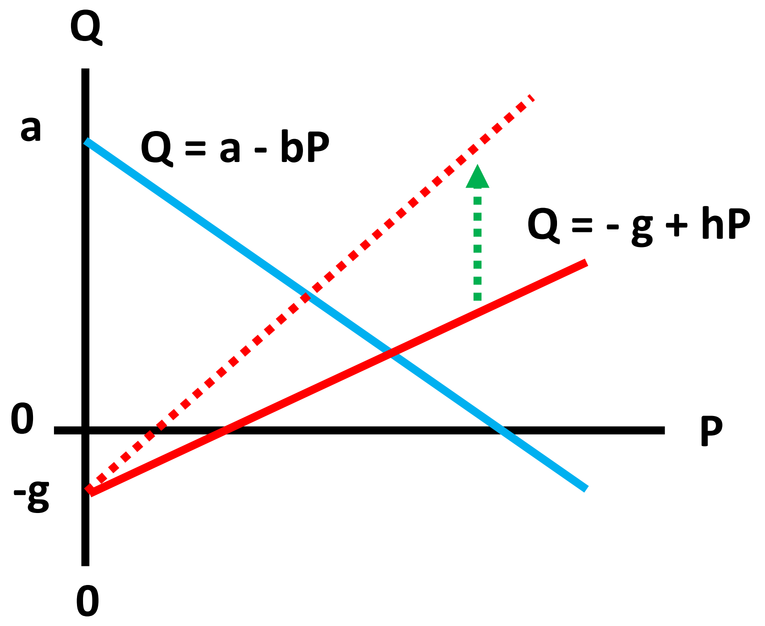


$$0 < z = \frac{Q^*}{Q^F} = \frac{b+h}{2b+h} < 1$$

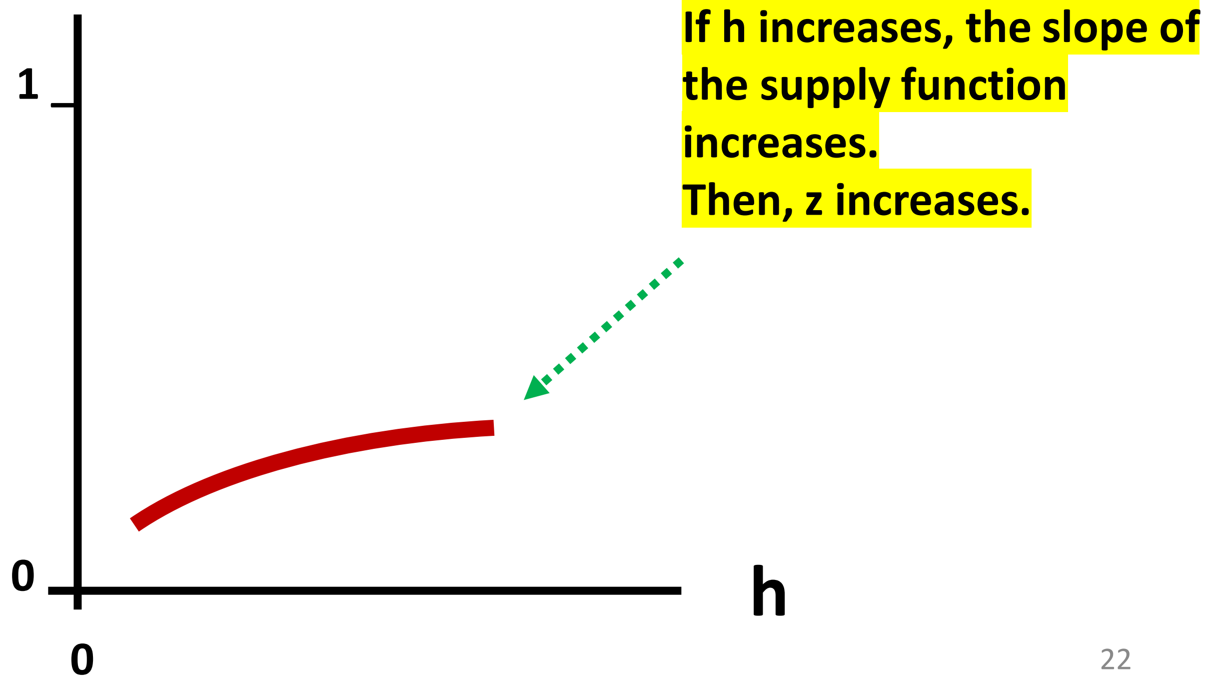
$$\frac{dz}{dh} = \frac{(1)(2b+h)-(b+h)(1)}{\left(2b+h\right)^2}$$

$$\frac{dz}{dh} = \frac{2b+h-b-h}{\left(2b+h\right)^2}$$

$$\frac{dz}{dh} = \frac{b}{\left(2b+h\right)^2} > 0$$



$$z = \frac{Q^*}{Q^F}$$



$$Q^* = \frac{ah - bg}{2b + h}$$ = *Optimal import level.*

$$P_H^* = \frac{a}{b} - \frac{1}{b}Q^*$$ = *Optimal price for the customers at home (H).*

(This is determined from the demand function.)

$$P_W^* = \frac{g}{h} + \frac{1}{h}Q^*$$ = *Optimal price for the exporters from the outside World (W).*

(This is determined from the supply function.)

$$T^* = P_H^* - P_W^*$$ = *Optimal Tariff.*

$$T^* = \left(\frac{a}{b} - \frac{1}{b}Q^*\right) - \left(\frac{g}{h} + \frac{1}{h}Q^*\right)$$

= *Optimal Tariff.*

$$T^* = \frac{a}{b} - \frac{1}{b}Q^* - \frac{g}{h} - \frac{1}{h}Q^*$$

$$T^* = \left(\frac{a}{b} - \frac{g}{h}\right) - \left(\frac{1}{b} + \frac{1}{h}\right)Q^*$$

$$T^* = \left(\frac{ah - bg}{bh}\right) - \left(\frac{b + h}{bh}\right)\frac{(ah - bg)}{(2b + h)}$$

$$T^* = \left(\frac{ah - bg}{bh}\right) - \left(\frac{b+h}{bh}\right)\frac{(ah-bg)}{(2b+h)}$$

$$T^* = \left(\frac{ah - bg}{bh}\right)\left(1 - \frac{(b+h)}{(2b+h)}\right)$$

$$T^* = \left(\frac{ah - bg}{bh}\right)\left(\frac{(2b+h)}{(2b+h)} - \frac{(b+h)}{(2b+h)}\right)$$

$$T^* = \left(\frac{ah - bg}{bh}\right)\left(\frac{2b+h-b-h}{2b+h}\right)$$

$$T^* = \left(\frac{ah - bg}{bh}\right)\left(\frac{b}{2b+h}\right)$$

The Optimal Tariff

$$T^* = \frac{ah - bg}{2bh + h^2}$$

The optimal tariff and the effects of parameter changes:

$$T^* = \frac{ah - bg}{2bh + h^2}$$

$$\frac{dT^*}{da} = \frac{1}{2b + h} > 0$$

$$\frac{dT^*}{db} = \frac{(-g)(2bh + h^2) - (ah - bg)(2h)}{(2bh + h^2)^2}$$

$$\frac{dT^*}{db} = \frac{-2bgh - gh^2 - 2ah^2 + 2bgh}{(2bh + h^2)^2}$$

$$\frac{dT^*}{db} = \frac{-(2a + g)h^2}{(2bh + h^2)^2} < 0$$

$$\frac{dT^*}{dg} = \frac{(-b)(2bh + h^2) - (2bh + h^2) \times 0}{(2bh + h^2)^2}$$

$$\frac{dT^*}{dg} = \frac{-b}{2bh + h^2} < 0$$

$$\frac{dT^*}{dh} = \frac{(a)(2bh + h^2) - (ah - bg)(2b + 2h)}{(2bh + h^2)^2}$$

$$\frac{dT^*}{dh} = \frac{2abh + ah^2 - 2abh - 2ah^2 + 2b^2 g + 2bgh}{(2bh + h^2)^2}$$

$$\frac{dT^*}{dh} = \frac{-ah^2 + 2b^2 g + 2bgh}{(2bh + h^2)^2}$$

?

$$\frac{dT^*}{dh} = \frac{-ah^2 + 2b^2 g + 2bgh}{\left(2bh + h^2\right)^2}$$

(h/b) very large: $$\frac{dT^*}{dh} > 0$$

(h/b) very small: $$\frac{dT^*}{dh} < 0$$

Concrete example:

a=8, b=1, g=4, h=2

Inverse demand function:

P = (a/b) – (1/b)Q

P = 8 - Q

Inverse supply function:

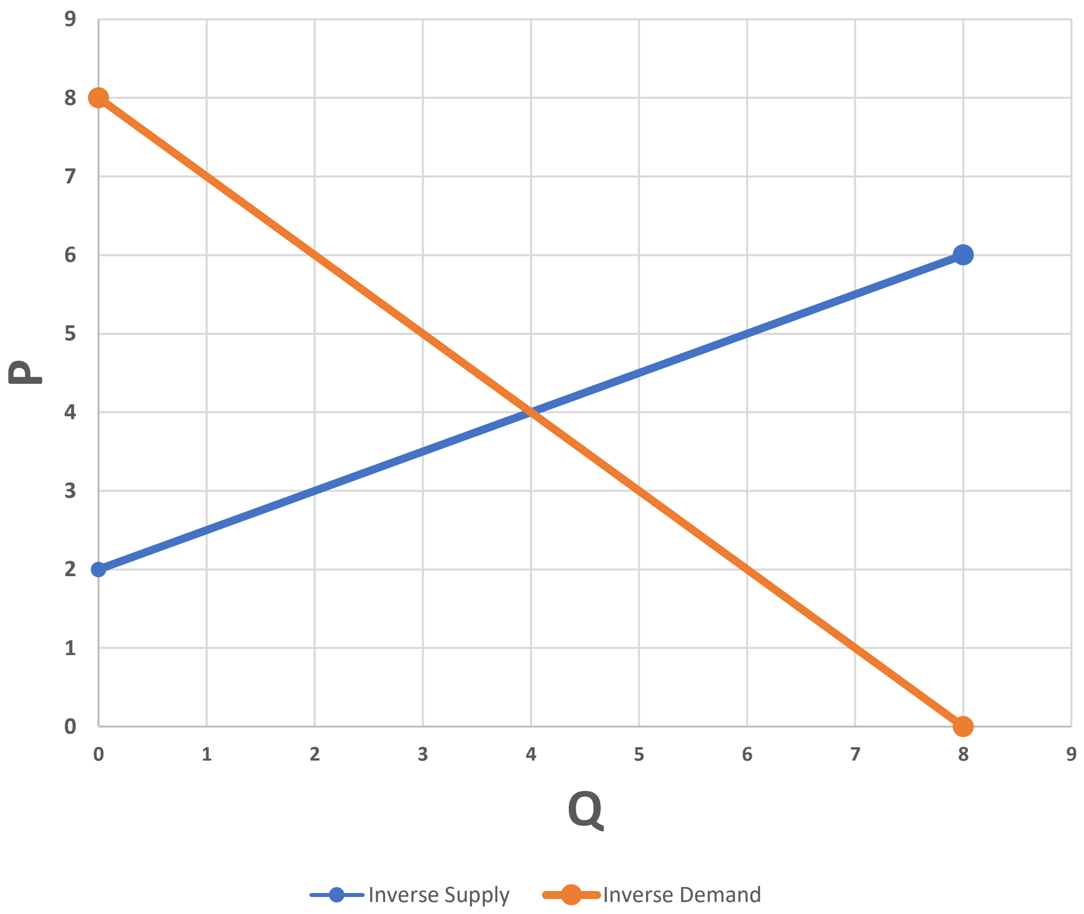
P = (g/h) – (1/h)Q

P = 2 + (1/2)Q

$$Q_D^F = \frac{ah - bg}{b + h} = \frac{16 - 4}{3} = 4$$

$$Q_S^F = \frac{ah - bg}{b + h} = 4$$

Free Trade Solution:



Concrete example:
a=8, b=1, g=4, h=2

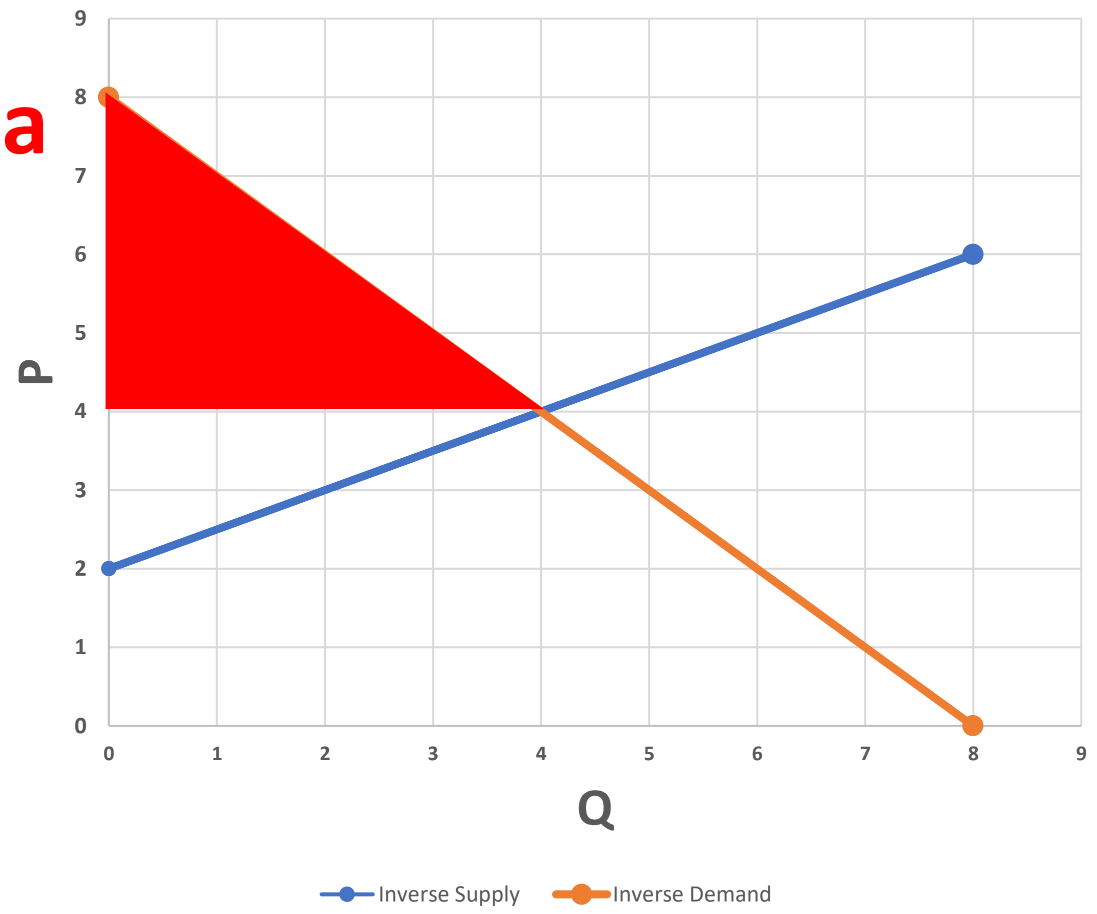
Free Trade Solution:

Consumer Surplus = Red Area
= (4*4)/2 = 8

$$S = \left(\frac{a}{b} - \frac{g}{h}\right)Q - \left(\frac{1}{2b} + \frac{1}{h}\right)Q^2$$

$$S = 6Q - Q^2$$

$$S = 6 \times 4 - (4)^2 = 8$$



Concrete example:

a=8, b=1, g=4, h=2

The optimal import level:

$$Q^* = \frac{ah - bg}{2b + h}$$

$$Q^* = \frac{16 - 4}{2 + 2} = \frac{12}{4} = 3$$

Optimization of the import level:

Here, we optimize the import volume.

A tariff is used to reduce the import volume, which also creates a tariff revenue.

The objective function is the sum of the consumer surplus and the tariff revenue.

This maximizes the total surplus of the consumers in the importing country N1 in case the tariff revenue later is redistributed to the consumers.

Observation:* *In the real world, it is quite possible that the tariff revenue is not completely redistributed to the consumers. In such cases, it is quite possible that the consumers do not benefit from the tariff, even if the total surplus is maximized via the tariff.

Concrete example:

a=8, b=1, g=4, h=2

The optimal price "at home" in N1:

$$P_H^* = \frac{a}{b} - \frac{1}{b}Q^* = 8 - 3 = 5$$

The optimal price "in the World market":

$$P_W^* = \frac{g}{h} + \frac{1}{h}Q^* = 2 + \left(\frac{1}{2}\right)3 = 3.5$$

Concrete example:

a=8, b=1, g=4, h=2

The optimal tariff:

$$T^* = \frac{ah - bg}{2bh + h^2}$$

$$T^* = \frac{16 - 4}{4 + 2^2} = \frac{12}{8} = 1.5$$

Concrete example:

a=8, b=1, g=4, h=2

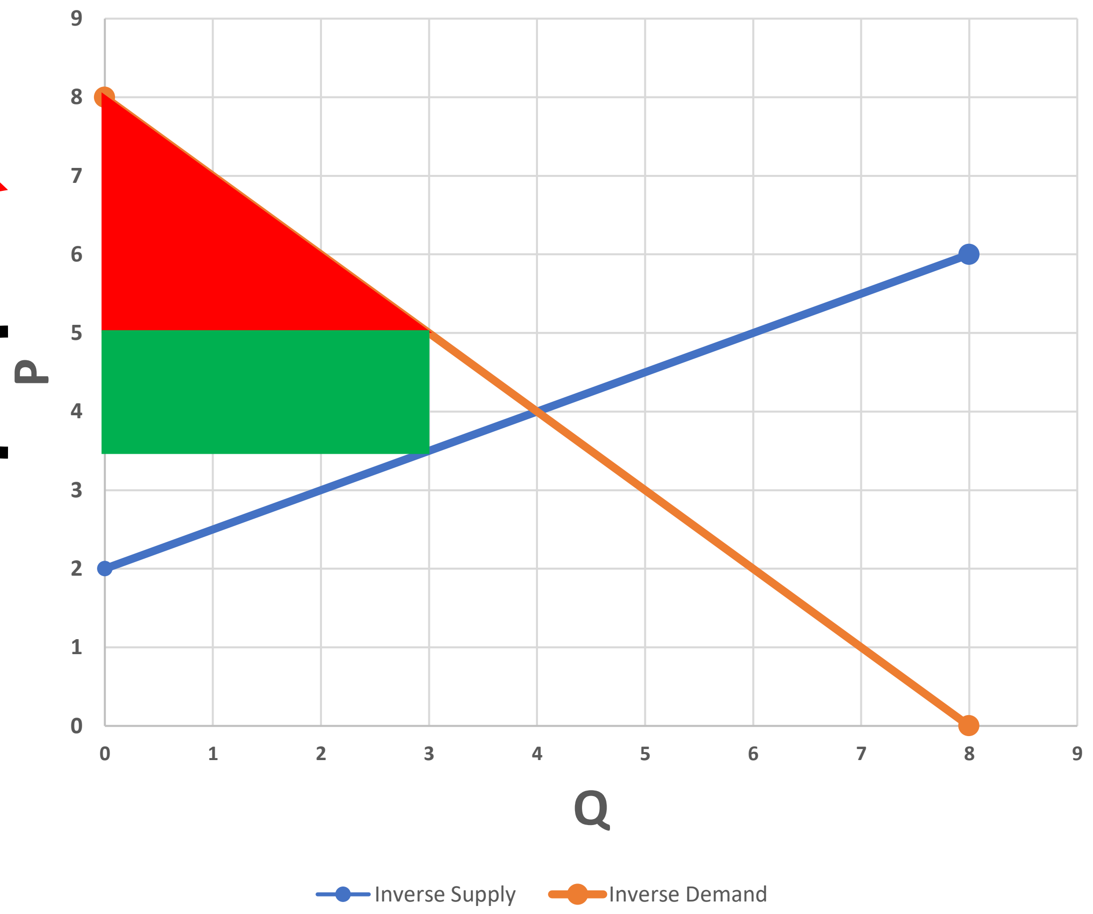
Optimal Tariff Solution:

Consumer Surplus = Red Area
= (3*3)/2 = 4.5

Tariff Gain = Green Area
= (3*1.5) = 4.5

Total Surplus = 9.

T^*



$$S=\left(\frac{a}{b}-\frac{g}{h}\right)Q-\left(\frac{1}{2b}+\frac{1}{h}\right)Q^2$$

$$S=6Q-Q^2$$

$$S=6\times3-(3)^2=9$$

Concrete example:
a=8, b=1, g=4, h=2

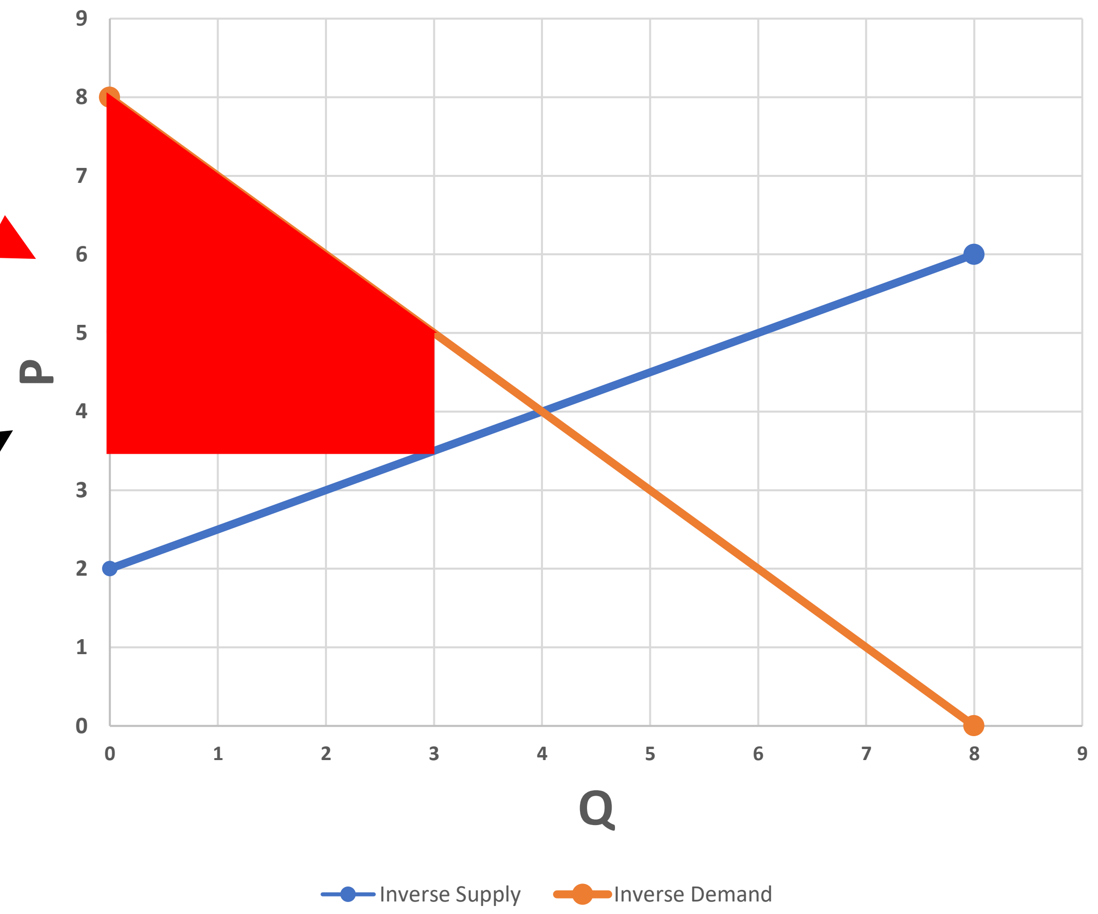
Alternative solution with the same total surplus, via a Consumer Cartel, *without* tariffs:

Consumer Surplus = Red Area
= (3*3)/2 + 1.5*3 = 9

$$S = \left(\frac{a}{b} - \frac{g}{h}\right)Q - \left(\frac{1}{2b} + \frac{1}{h}\right)Q^2$$

$$S = 6Q - Q^2$$

$$S = 6 \times 3 - (3)^2 = 9$$



Some results from the concrete example:

The total surplus in N1 increases from 8 to 9, when the free trade solution is replaced by the optimized solution. (The consumer surplus is reduced from 8 to 4.5 and the tariff revenue increases from 0 to 4.5.)

Observation:
In the real world, it is quite possible that the tariff revenue is not completely redistributed to the consumers.
In such cases, it is quite possible that the consumers do not benefit from the tariff,
even if the total surplus is maximized via the tariff.

Free Trade Equilibrium Price = 4

Optimal Tariff = 1.5
World Price in new equilibrium when the optimal tariff is applied = 3.5
Price in N1 when the optimal tariff is applied = 5

The Optimal Tariff = 37.5 % of the Free Trade Equilibrium Price
The Optimal Tariff = 42.9 % of World Price in new equilibrium when the optimal tariff is applied.

Abstract part 1

Linear and nonlinear optimization of international trade and tariffs, WSTA 2025, Peter Lohmander

- **International trade, with and without tariffs, is analytically and numerically defined and analyzed. First, the linear partial equilibrium two nation trade model approach by Pindyck and Rubinfeld and by Krugman and Obstfeld, is applied. The producer and consumer surpluses are determined in the two nations, N1 and N2, with and without trade. With free trade without tariffs, the total surpluses in the two nations, including producer and consumer surpluses, exceed the total surpluses in the two countries, without trade.**

Results from the linear, partial equilibrium, model

Tariff Results by Peter Lohmander

Solutions without trade:

Country	Price	Production	Consumption
1.000	7.000	3.000	3.000
2.000	3.500	2.500	2.500

Country	1	2
Consumer surplus	4.500	3.125
Pmin1 and Pmin2	4.000	1.000
Producer surplus	4.500	3.125
Total surplus	9.000	6.250

Total surplus World without trade 15.250

All parameters, equations, and the computer code are found in the Appendix.

Results from the linear, partial equilibrium, model

```
Solutions with free trade without tariffs:

Free trade Price =  5.25

 Country          Price   Production  Consumption
   1.000          5.250        1.250        4.750
   2.000          5.250        4.250        0.750

Exp1F and Exp2F =      -3.500      3.500

Country                      1          2
Consumer surplus        11.281      0.281
Pmin1 and Pmin2          4.000      1.000
Producer surplus         0.781      9.031
Total surplus           12.063      9.313

Total surplus World with free trade   21.375
```

All parameters, equations, and the computer code are found in the Appendix.

Abstract part 2

Linear and nonlinear optimization of international trade and tariffs, WSTA 2025, Peter Lohmander

- **Then, N1 introduces tariffs on imports. The objective function of N1 is the total surplus of N1, including consumer and producer surplus and the tariff gain. The optimal tariff for N1 is determined as a general function of all parameters of demand and supply in the two nations. The objective function of N1 is a strictly concave and quadratic function of the tariff level, which implies that the optimal tariff is unique and instantly determined from the first order optimum condition.**

Abstract part 3

Linear and nonlinear optimization of international trade and tariffs, WSTA 2025, Peter Lohmander

- **. In N1, it is proved that the optimal tariff is strictly positive. In N1, with the optimal tariff, the tariff gain and the producer surplus increase and the consumer surplus decreases, compared to the free trade solution. In N2, the consumer surplus increases and the producer surplus decreases, if the tariff in N1 increases. The total surplus in N2 however decreases. With a strictly positive N1 tariff, the total surplus in both nations, is reduced, compared to the free trade solution.**

Results from the linear, partial equilibrium, model

```
Analytical results:
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topt =      1.167
resopt =      1.021

Solution with trade and tariff optimized by Country 1:

Optimized Tariff =      1.167

dSUR1, dSUR2, dSURW =      1.021    -1.701    -0.681

 Country             Price    Production   Consumption
   1.000             5.833         1.833         4.167
   2.000             4.667         3.667         1.333

Exp1t and Exp2t             -2.333       2.333

Country                          1           2
Consumer surplus             8.681       0.889
Producer surplus             1.681       6.722
Tariff gain                  2.722
Total surplus               13.083       7.611
Total surplus World with optimal tariff          20.694
```

Tariff optimized by N1.

Thanks to the optimized tariff, the total surplus of N1 increases by this amount.

All parameters, equations, and the computer code are found in the Appendix.

Nonlinear analysis

Utility

Consumption levels

$$\max U_i = U_i\left(c_{i,1},..,c_{i,\eta}\right) = \kappa_i c_{i,1}{}^{\alpha_{i,1}} \times \ldots \times c_{i,\eta}{}^{\alpha_{i,\eta}}$$

$$\sum_{j=1}^{\eta} \alpha_{i,j} = 1$$

Consumption budget constraint

$$p_1 c_{i,1} + \ldots + p_\eta c_{i,\eta} \le B_i$$

Lagrange function

$$L_i = \kappa_i c_{i,1}{}^{\alpha_{i,1}} \times \ldots \times c_{i,\eta}{}^{\alpha_{i,\eta}} + \lambda_i \left(B_i - p_1 c_{i,1} - \ldots - p_\eta c_{i,\eta} \right)$$

$$\frac{dL_i}{d\lambda_i} = B_i - p_1 c_{i,1} - \ldots - p_\eta c_{i,\eta}$$

$$\frac{dL_i}{dc_{i,1}} = \frac{\alpha_{i,1} U_i}{c_{i,1}} - p_1 \lambda_i$$

$$\frac{dL_i}{dc_{i,\eta}} = \frac{\alpha_{i,\eta} U_i}{c_{i,\eta}} - p_\eta \lambda_i$$

First order derivatives

$$\begin{cases} \dfrac{dL_i}{d\lambda_i} = B_i - p_1 c_{i,1} - \ldots - p_\eta c_{i,\eta} = 0 \\ \dfrac{dL_i}{dc_{i,1}} = \dfrac{\alpha_{i,1} U_i}{c_{i,1}} - p_1 \lambda_i = 0 \\ \bullet \\ \dfrac{dL_i}{dc_{i,\eta}} = \dfrac{\alpha_{i,\eta} U_i}{c_{i,\eta}} - p_\eta \lambda_i = 0 \end{cases}$$

First order optimum conditions of interior solution

$$\frac{dL_i}{dc_{i,j}} = \frac{\alpha_{i,j} U_i}{c_{i,j}} - p_j \lambda_i \quad \forall i \in \{1,...,I\}, j \in \{1,...,\eta\}$$

$$\left(\frac{dL_i}{dc_{i,j}} = 0 \right) \Rightarrow \left(\frac{\alpha_{i,j} U_i}{c_{i,j}} - p_j \lambda_i = 0 \right)$$

$$\frac{\alpha_{i,j} U_i}{c_{i,j}} = p_j \lambda_i$$

$$c_{i,j} = \frac{\alpha_{i,j} U_i}{p_j \lambda_i}$$

$$B_i = \sum_{j=1}^{\eta} p_j c_{i,j}$$

$$B_i = \sum_{j=1}^{\eta} p_j \left(\frac{\alpha_{i,j} U_i}{p_j \lambda_i} \right)$$

$$B_i = \sum_{j=1}^{\eta} \frac{\alpha_{i,j} U_i}{\lambda_i}$$

$$B_i = \frac{U_i}{\lambda_i} \sum_{j=1}^{\eta} \alpha_{i,j}$$

$$\frac{U_i}{\lambda_i} = B_i \left(\sum_{j=1}^{\eta} \alpha_{i,j} \right)^{-1} = B_i \quad for \quad \sum_{j=1}^{\eta} \alpha_{i,j} = 1$$

$$c_{i,j} = \frac{B_i \alpha_{i,j}}{p_j}, \quad \forall i, j$$

Observation:

The optimal demand (or consumption) function, in nation i, for product j, is proportional to the consumption budget and to the respective utility function exponent.

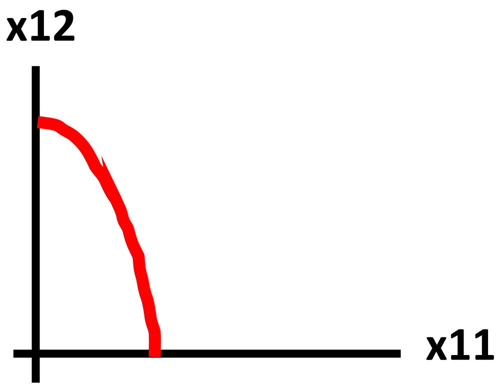
It is inversely proportional to the product price.

A first simple nonlinear analysis with two countries, N1 and N2.

The production volumes of product 1 and 2, produced in N1, are x_{11} and x_{12}.

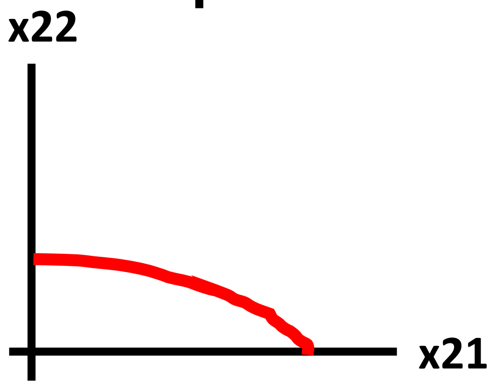
The production volumes of product 1 and 2, produced in N2, are x_{21} and x_{22}.

In N1, the production possibility frontier is:



$$x_{12} = 4 - (x_{11})^2 \quad , \quad x_{11} \geq 0, x_{12} \geq 0$$

In N2, the corresponding production possibility frontier is:



$$x_{21} = 4 - (x_{22})^2 \quad , \quad x_{21} \geq 0, x_{22} \geq 0$$

Production optimization in N1:

$$\max_{x_{11}} \pi_1 = p_1 x_{11} + p_2 x_{12}$$

$$s.t. \quad x_{12} = 4 - (x_{11})^2 \quad , \quad x_{11} \geq 0, x_{12} \geq 0$$

$$\max_{x_{11}} \pi_1 = p_1 x_{11} + p_2 \left(4 - (x_{11})^2\right)$$

$$\max_{x_{11}} \pi_1 = p_1 x_{11} + 4 p_2 - (x_{11})^2 p_2$$

$$\frac{d\pi_1}{dx_{11}} = p_1 - 2x_{11}p_2 = 0$$

$$\frac{d^2\pi_1}{dx_{11}{}^2} = -2p_2 < 0 \quad , \quad p_2 > 0$$

$$\left(\frac{d\pi_1}{dx_{11}} = 0 \right) \Rightarrow x_{11}{}^* = \frac{1}{2}\left(\frac{p_1}{p_2} \right)$$

$$\left(r = \frac{p_1}{p_2} \right) \Rightarrow x_{11}{}^* = \frac{1}{2} r$$

$$x_{12}{}^* = 4 - \left(x_{11}{}^* \right)^2$$

$$x_{12}{}^* = 4 - \frac{1}{4} r^2$$

We note that the optimal production levels in N1 are functions of the relative price r.

Production optimization in N2:

$$\max_{x_{22}} \pi_2 = p_1 x_{21} + p_2 x_{22}$$

$$s.t. \quad x_{21} = 4 - \left(x_{22}\right)^2 \quad , \quad x_{21} \geq 0, x_{22} \geq 0$$

$$\max_{x_{22}} \pi_2 = p_1 \left(4 - \left(x_{22}\right)^2\right) + p_2 x_{22}$$

$$\max_{x_{22}} \pi_2 = p_1 \left(4 - x_{22}{}^2\right) + 4 p_2 x_{22}$$

$$\frac{d\pi_2}{dx_{22}} = p_2 - 2x_{22}p_1 = 0$$

$$\frac{d^2\pi_2}{dx_{22}^{\ 2}} = -2p_1 < 0 \quad , \quad p_1 > 0$$

$$\left(\frac{d\pi_2}{dx_{22}} = 0\right) \Rightarrow x_{22}^{\ *} = \frac{1}{2}\left(\frac{p_2}{p_1}\right)$$

$$\left(r = \frac{p_1}{p_2}\right) \Rightarrow x_{22}{}^* = \frac{1}{2} r^{-1}$$

$$x_{21}{}^* = 4 - \left(x_{22}{}^*\right)^2$$

$$x_{21}{}^* = 4 - \frac{1}{4} r^{-2}$$

We note that also the optimal production levels in N2 are functions of the relative price r.

$$x_{11}{}^{*} = \frac{1}{2} r$$

$$x_{12}{}^{*} = 4 - \frac{1}{4} r^{2}$$

$$x_{21}{}^{*} = 4 - \frac{1}{4} r^{-2}$$

$$x_{22}{}^{*} = \frac{1}{2} r^{-1}$$

The optimal production levels in both countries, are explicit functions of the relative price, r.

```
! First illustration;
! Utility maximization without trade;

max = u1;

u1 = c11^a*c12^a;
a = 1/2;

x12 = 4 - x11^2;

c11 = x11;
c12 = x12;

end
```

Variable	Value
C11	1.154701
X11	1.154701
C12	2.666667
X12	2.666667
A	0.5000000
U1	1.754765

```
! Second illustration:
! Free trade;

a = 1/2;
u1 = c11^a*c12^a;
u2 = c21^a*c22^a;

x12 = 4 - x11^2;
x21 = 4 - x22^2;

x11 = 1/2*r;
x21 = 4-1/4*r^(-2);

B1 = p1*x11 + p2*x12;
B2 = p1*x21 + p2*x22;

c11 = a*B1/p1;
c12 = a*B1/p2;

c21 = a*B2/p1;
c22 = a*B2/p2;

s11 = x11 - c11;    <- Excess supply of prod 1 from N1
s12 = x12 - c12;    <- Excess supply of prod 2 from N1

s21 = x21 - c21;    <- Excess supply of prod 1 from N2
s22 = x22 - c22;    <- Excess supply of prod 2 from N2

@free(s11);
@free(s12);
@free(s21);
@free(s22);

s1 = s11 + s21;     <- World excess supply of prod 1
s2 = s12 + s22;     <- World excess supply of prod 2
r = p1/p2;
p1 = 1;
s1 = 0;             <- World excess supply (of prod 1) = 0.

end
```

Variable	Value
A	0.5000000
U1	2.124999
C11	2.124998
C12	2.125000
U2	2.125004
C21	2.125003
C22	2.125004
X12	3.750000
X11	0.5000003
X21	3.750001
X22	0.5000045
R	1.000001
B1	4.249996
P1	1.000000
P2	0.9999989
B2	4.250006
S11	-1.624998
S12	1.625000
S21	1.624998
S22	-1.624999
S1	0.000000
S2	0.2466676E-06

Utility levels in free trade

```
! Third illustration:
! Trade with tariff;
T = 0.0;
TariffRev = T*(-s11);
a = 1/2;

u1 = c11^a*c12^a;
u2 = c21^a*c22^a;
usum = u1 + u2;

x12 = 4 - x11^2;
x21 = 4 - x22^2;
x11 = 1/2*z;
x21 = 4-1/4*r^(-2);

[_B1] B1 = p3*x11 + p2*x12 + TariffRev;
[_B2] B2 = p1*x21 + p2*x22;

c11 = a*B1/p3;
c12 = a*B1/p2;
c21 = a*B2/p1;
c22 = a*B2/p2;

s11 = x11 - c11;
s12 = x12 - c12;
s21 = x21 - c21;
s22 = x22 - c22;

@free(s11);
@free(s12);
@free(s21);
@free(s22);
@free(s1);
@free(s2);
@free(TariffRev);

s1 = s11 + s21;
s2 = s12 + s22;

r = p1/p2;
p3 = p1*(1+T);
z = p3/p2;
p1 = 1;

s1 = 0;

end
```

Variable	Value
T	0.000000
TARIFFREV	0.000000
S11	-1.624998
A	0.5000000
U1	2.124999
C11	2.124999
C12	2.125000
U2	2.125002
C21	2.125002
C22	2.125002
USUM	4.250002
X12	3.750000
X11	0.5000002
X21	3.750000
X22	0.5000027
Z	1.000000
R	1.000000
B1	4.249997
P3	1.000000
P2	0.9999993
B2	4.250004
P1	1.000000
S12	1.625000
S21	1.624998
S22	-1.625000
S1	0.000000
S2	0.2115638E-06

T (%)	u1	u2	usum
0	2.125	2.125	4.250
50	2.244	1.971	4.216
100	2.262	1.883	4.145
150	2.229	1.830	4.059

Alternative way to calculate the effects of the tariff based on iterative determination of the tariff revenue and the excess supply functions

The software is developed in Smart Basic.

Everything is found in the Appendix.

Abstract part 4

Linear and nonlinear optimization of international trade and tariffs, WSTA 2025, Peter Lohmander

- **Then, a nonlinear general equilibrium trade model with three nations and four products is defined and analyzed. Each nation, N1, N2 and N3, can produce two different products. The country specific production possibility frontiers are strictly concave and guarantee strictly positive production levels for all feasible relative prices. The demand functions in the different countries are determined from maximized nonlinear Cobb Douglas utility functions with country specific parameters and the national consumption budgets. These budgets are functions of all relative prices, zero excess supplies in all international product markets, and optimized production levels in all countries.**

A nonlinear general equilibrium trade model with three nations and four products is defined and analyzed.

Each nation, N1, N2 and N3, can produce two different products.
N1 can produce products x1 and x2. N2 can produce x2 and x3. N3 can produce x3 and x4.

The country specific production possibility frontiers are strictly concave and guarantee strictly positive production levels for all feasible relative prices.

Below, j and k are two different product indices. These indices represent different products in the different countries.

Here is the production possibility frontier of nation i:

$$m_{ij}\left(x_{ij}\right)^2 + m_{ik}\left(x_{ik}\right)^2 = 1$$

$$x_{ik} = \sqrt{m_{ik}^{-1}\left(1 - m_{ij}\left(x_{ij}\right)^2\right)}$$

Profit maximization in each country occurs based on the relative prices.

$$\max \pi_i = p_j x_{ij} + p_k x_{ik}$$

$$\max \pi_i = p_j x_{ij} + p_k \left(m_{ik}{}^{-1} \left(1 - m_{ij} \left(x_{ij} \right)^2 \right) \right)^{\frac{1}{2}}$$

The first order optimum condition is:

$$\frac{d\pi_i}{dx_{ij}} = p_j + \frac{p_k}{2} \left(m_{ik}{}^{-1} \left(1 - m_{ij} \left(x_{ij} \right)^2 \right) \right)^{-\left(\frac{1}{2}\right)} \times (-2) \frac{m_{ij}}{m_{ik}} x_{ij} = 0$$

$$\frac{d\pi_i}{dx_{ij}} = p_j - p_k\left(m_{ik}^{-1}\left(1-m_{ij}\left(x_{ij}\right)^2\right)\right)^{-\left(\frac{1}{2}\right)}\left(\frac{m_{ij}}{m_{ik}}\right)x_{ij} = 0$$

$$\frac{d^2\pi_i}{dx_{ij}^{\ 2}} = \left(\frac{1}{2}\right)p_k\left(m_{ik}^{-1}\left(1-m_{ij}\left(x_{ij}\right)^2\right)\right)^{-\left(\frac{3}{2}\right)}\left(-2m_{ik}^{-1}m_{ij}x_{ij}\right)\left(\frac{m_{ij}}{m_{ik}}\right)x_{ij} - p_k\left(m_{ik}^{-1}\left(1-m_{ij}\left(x_{ij}\right)^2\right)\right)^{-\left(\frac{1}{2}\right)}\left(\frac{m_{ij}}{m_{ik}}\right)$$

$$\frac{d^2\pi_i}{dx_{ij}^{\ 2}} = -\left[p_k\frac{m_{ij}}{m_{ik}}\right]\left(\left(m_{ik}^{-1}\left(1-m_{ij}\left(x_{ij}\right)^2\right)\right)^{-\left(\frac{3}{2}\right)}\left(\frac{m_{ij}}{m_{ik}}\right)\left(x_{ij}\right)^2 + \left(m_{ik}^{-1}\left(1-m_{ij}\left(x_{ij}\right)^2\right)\right)^{-\left(\frac{1}{2}\right)}\right) < 0$$

Hence, the profit function is strictly concave with a unique maximum.

The first order optimum condition can be used to derive the optimal production decisions.

$$\left(\frac{d\pi_i}{dx_{ij}}=0\right)\Rightarrow\left(p_k\left(m_{ik}{}^{-1}\left(1-m_{ij}\left(x_{ij}\right)^2\right)\right)^{-\left(\frac{1}{2}\right)}\left(\frac{m_{ij}}{m_{ik}}\right)x_{ij}=p_j\right)$$

$$\left(m_{ik}{}^{-1}\left(1-m_{ij}\left(x_{ij}\right)^2\right)\right)^{-\left(\frac{1}{2}\right)}\left(\frac{m_{ij}}{m_{ik}}\right)x_{ij}=\frac{p_j}{p_k}$$

(Many steps are omitted here.)

$$x_{ij}=\sqrt{\frac{1}{m_{ij}\left[\frac{m_{ij}}{m_{ik}}\left(\frac{p_k}{p_j}\right)^2+1\right]}}$$

$$x_{ik}=\sqrt{m_{ik}{}^{-1}-\frac{1}{\left[m_{ij}\left(\frac{p_k}{p_j}\right)^2+m_{ik}\right]}}$$

$$p_R = \frac{p_j}{p_k}$$

Then, the production levels are the following explicit functions of this relative price:

$$x_{ij} = \sqrt{\frac{1}{m_{ij}\left[\frac{m_{ij}}{m_{ik}} p_R^{-1} + 1\right]}}$$

$$x_{ik} = \sqrt{m_{ik}^{-1} - \frac{1}{\left[m_{ij} p_R^{-1} + m_{ik}\right]}}$$

Abstract part 5

Linear and nonlinear optimization of international trade and tariffs, WSTA 2025, Peter Lohmander

- **Three equilibrium relative prices are numerically determined from the total nonlinear system. These prices guarantee that the total excess supply functions approach zero in all international product markets. Rapid and reliable relative price convergence is obtained via new and generalized multivariate bisection method in three dimensions. In this process, the objective function is defined as the sum of squares of all excess supply function values.**

All parameters, equations, and the computer code are found in the Appendix.

Nonlinear Model: Examples of Results and Software

Results and software by Peter Lohmander
2025–06-02

Case 0

Tariff (%) = 0

Production possibility frontiers parameters

----

x11max, x12max, x13max, x14max = 2.000 4.000 0.000 0.000
x21max, x22max, x23max, x24max = 0.000 2.000 4.000 0.000
x31max, x32max, x33max, x34max = 0.000 0.000 2.000 4.000

Total sum of squared excess supplies = tesopt = 6.456329543206307D-08

Equilibrium prices = p1, p2, p3, p4 = 100.000 45.449 77.129 95.391

Global Excess Supplies = es1, es2, es3, es4 = 0.00019 -0.00010 -0.00005 -0.00012

Production levels =

x11, x12, x13, x14 = 1.480 2.691 0.000 0.000
x21, x22, x23, x24 = 0.000 0.565 3.837 0.000
x31, x32, x33, x34 = 0.000 0.000 0.750 3.708

Excess supply levels =

s11, s12, s13, s14 = 0.804 1.204 -0.876 -0.708
s21, s22, s23, s24 = -0.804 -1.204 2.794 -0.843
s31, s32, s33, s34 = 0.000 0.000 -1.918 1.551

Consumption levels =

c11, c12, c13, c14 = 0.676 1.487 0.876 0.708
c21, c22, c22, c24 = 0.804 1.769 1.043 0.843
c31, c32, c33, c34 = 0.000 0.000 2.668 2.157

GNP levels = w1, w2 ,w3 = 270.278 321.628 411.565

Utility levels = u1, u2, u3 = 0.889 1.115 2.399

Global GNP = 1003.470
Global Utility = 4.403

Case 0

Tariff (%) = 50

Production possibility frontiers parameters

----

x11max, x12max, x13max, x14max =	2.000	4.000	0.000	0.000
x21max, x22max, x23max, x24max =	0.000	2.000	4.000	0.000
x31max, x32max, x33max, x34max =	0.000	0.000	2.000	4.000

Total sum of squared excess supplies = tesopt = 3.99792725031482D-08

Equilibrium prices = p1, p2, p3, p4 = 100.000 45.176 71.396 82.832

Global Excess Supplies = es1, es2, es3, es4 = 0.00003 0.00003 0.00012 -0.00015

Production levels =

x11, x12, x13, x14 =	1.484	2.682	0.000	0.000
x21, x22, x23, x24 =	0.000	0.603	3.814	0.000
x31, x32, x33, x34 =	0.000	0.000	0.792	3.673

Excess supply levels =

s11, s12, s13, s14 =	0.749	1.054	-1.030	-0.592
s21, s22, s23, s24 =	-0.749	-1.054	2.765	-0.904
s31, s32, s33, s34 =	0.000	0.000	-1.735	1.496

Consumption levels =

c11, c12, c13, c14 =	0.735	1.627	1.030	0.592
c21, c22, c22, c24 =	0.749	1.658	1.049	0.904
c31, c32, c33, c34 =	0.000	0.000	2.527	2.178

GNP levels = w1, w2 ,w3 = 294.047 299.538 360.788

Utility levels = u1, u2, u3 = 0.924 1.081 2.346

Global GNP = 954.373
Global Utility = 4.351

Case 0

Tariff (%) = 100

Production possibility frontiers parameters

----

x11max, x12max, x13max, x14max =	2.000	4.000	0.000	0.000
x21max, x22max, x23max, x24max =	0.000	2.000	4.000	0.000
x31max, x32max, x33max, x34max =	0.000	0.000	2.000	4.000

Total sum of squared excess supplies = tesopt = 8.681383082576513D-08

Equilibrium prices = p1, p2, p3, p4 = 100.000 45.000 68.145 75.742

Global Excess Supplies = es1, es2, es3, es4 = 0.00018 -0.00021 -0.00001 -0.00011

Production levels =

x11, x12, x13, x14 =	1.487	2.676	0.000	0.000
x21, x22, x23, x24 =	0.000	0.627	3.798	0.000
x31, x32, x33, x34 =	0.000	0.000	0.820	3.648

Excess supply levels =

s11, s12, s13, s14 =	0.718	0.967	-1.128	-0.507
s21, s22, s23, s24 =	-0.718	-0.968	2.745	-0.947
s31, s32, s33, s34 =	0.000	0.000	-1.617	1.455

Consumption levels =

c11, c12, c13, c14 =	0.769	1.708	1.128	0.507
c21, c22, c22, c24 =	0.718	1.595	1.053	0.947
c31, c32, c33, c34 =	0.000	0.000	2.438	2.193

GNP levels = w1, w2 ,w3 = 307.511 287.052 332.212

Utility levels = u1, u2, u3 = 0.931 1.061 2.312

Global GNP = 926.775
Global Utility = 4.305

"MAXIMUM" for N1

Case 0

Tariff (%) = 150

Production possibility frontiers parameters
----
x11max, x12max, x13max, x14max = 2.000 4.000 0.000 0.000
x21max, x22max, x23max, x24max = 0.000 2.000 4.000 0.000
x31max, x32max, x33max, x34max = 0.000 0.000 2.000 4.000

Total sum of squared excess supplies = tesopt = 1.04538358161312D-07

Equilibrium prices = p1, p2, p3, p4 = 100.000 44.893 66.074 71.221

Global Excess Supplies = es1, es2, es3, es4 = -0.00020 0.00023 0.00002 0.00011

Production levels =

x11, x12, x13, x14 = 1.488 2.672 0.000 0.000
x21, x22, x23, x24 = 0.000 0.643 3.787 0.000
x31, x32, x33, x34 = 0.000 0.000 0.842 3.629

Excess supply levels =

s11, s12, s13, s14 = 0.698 0.911 -1.196 -0.444
s21, s22, s23, s24 = -0.698 -0.911 2.731 -0.980
s31, s32, s33, s34 = 0.000 0.000 -1.535 1.424

Consumption levels =

c11, c12, c13, c14 = 0.791 1.761 1.196 0.444
c21, c22, c22, c24 = 0.698 1.554 1.056 0.980
c31, c32, c33, c34 = 0.000 0.000 2.376 2.205

GNP levels = w1, w2 ,w3 = 316.218 279.131 314.040

Utility levels = u1, u2, u3 = 0.927 1.049 2.289

Global GNP = 909.390
Global Utility = 4.265

Case 0

Tariff (%) = 200

Production possibility frontiers parameters

----

x11max, x12max, x13max, x14max =	2.000	4.000	0.000	0.000
x21max, x22max, x23max, x24max =	0.000	2.000	4.000	0.000
x31max, x32max, x33max, x34max =	0.000	0.000	2.000	4.000

Total sum of squared excess supplies = tesopt = 4.90099160193288D-08

Equilibrium prices = p1, p2, p3, p4 = 100.000 44.805 64.619 68.057

Global Excess Supplies = es1, es2, es3, es4 = -0.00010 -0.00006 0.00019 -0.00000

Production levels =

x11, x12, x13, x14 =	1.489	2.669	0.000	0.000
x21, x22, x23, x24 =	0.000	0.655	3.779	0.000
x31, x32, x33, x34 =	0.000	0.000	0.858	3.613

Excess supply levels =

s11, s12, s13, s14 =	0.684	0.871	-1.247	-0.395
s21, s22, s23, s24 =	-0.684	-0.871	2.721	-1.005
s31, s32, s33, s34 =	0.000	0.000	-1.474	1.400

Consumption levels =

c11, c12, c13, c14 =	0.806	1.798	1.247	0.395
c21, c22, c22, c24 =	0.684	1.526	1.058	1.005
c31, c32, c33, c34 =	0.000	0.000	2.332	2.214

GNP levels = w1, w2 ,w3 = 322.261 273.569 301.347

Utility levels = u1, u2, u3 = 0.919 1.040 2.272

Global GNP = 897.176
Global Utility = 4.231

In N2, the production possibility frontier changes.

Case 1 (x22max and x23max have been changed)

Tariff (%) = 0

Production possibility frontiers parameters

----

x11max, x12max, x13max, x14max =	2.000	4.000	0.000	0.000
x21max, x22max, x23max, x24max =	0.000	4.000	2.000	0.000
x31max, x32max, x33max, x34max =	0.000	0.000	2.000	4.000

Total sum of squared excess supplies = tesopt = 1.031094533439831D-08

Equilibrium prices = p1, p2, p3, p4 = 100.000 38.730 173.213 144.229

Global Excess Supplies = es1, es2, es3, es4 = -0.00005 -0.00006 -0.00001 0.00007

Production levels =

x11, x12, x13, x14 =	1.581	2.450	0.000	0.000
x21, x22, x23, x24 =	0.000	1.633	1.826	0.000
x31, x32, x33, x34 =	0.000	0.000	1.030	3.429

Excess supply levels =

s11, s12, s13, s14 =	0.949	0.817	-0.365	-0.439
s21, s22, s23, s24 =	-0.949	-0.817	1.278	-0.658
s31, s32, s33, s34 =	0.000	0.000	-0.913	1.096

Consumption levels =

c11, c12, c13, c14 =	0.632	1.633	0.365	0.439
c21, c22, c22, c24 =	0.949	2.450	0.548	0.658
c31, c32, c33, c34 =	0.000	0.000	1.943	2.333

GNP levels = w1, w2 ,w3 = 252.984 379.489 672.934

Utility levels = u1, u2, u3 = 0.638 0.914 2.129

Global GNP = 1305.407

Global Utility = 3.680

Case 1 (x22max and x23max have been changed)

Tariff (%) = 50

Production possibility frontiers parameters
----
x11max, x12max, x13max, x14max = 2.000 4.000 0.000 0.000
x21max, x22max, x23max, x24max = 0.000 4.000 2.000 0.000
x31max, x32max, x33max, x34max = 0.000 0.000 2.000 4.000

Total sum of squared excess supplies = tesopt = 7.515735279736754D-08

Equilibrium prices = p1, p2, p3, p4 = 100.000 38.340 162.744 130.654

Global Excess Supplies = es1, es2, es3, es4 = 0.00024 -0.00008 -0.00010 -0.00003

Production levels =

x11, x12, x13, x14 = 1.587 2.434 0.000 0.000
x21, x22, x23, x24 = 0.000 1.705 1.809 0.000
x31, x32, x33, x34 = 0.000 0.000 1.057 3.395

Excess supply levels =

s11, s12, s13, s14 = 0.900 0.641 -0.422 -0.351
s21, s22, s23, s24 = -0.900 -0.641 1.257 -0.688
s31, s32, s33, s34 = 0.000 0.000 -0.834 1.039

Consumption levels =

c11, c12, c13, c14 = 0.687 1.793 0.422 0.351
c21, c22, c22, c24 = 0.900 2.346 0.553 0.688
c31, c32, c33, c34 = 0.000 0.000 1.892 2.356

GNP levels = w1, w2 ,w3 = 274.942 359.808 615.688

Utility levels = u1, u2, u3 = 0.654 0.896 2.111

Global GNP = 1250.437
Global Utility = 3.661

"MAXIMUM" for N1

Case 1 (x22max and x23max have been changed)

Tariff (%) = 100

Production possibility frontiers parameters

----

x11max, x12max, x13max, x14max = 2.000 4.000 0.000 0.000
x21max, x22max, x23max, x24max = 0.000 4.000 2.000 0.000
x31max, x32max, x33max, x34max = 0.000 0.000 2.000 4.000

Total sum of squared excess supplies = tesopt = 1.027059118454071D-08

Equilibrium prices = p1, p2, p3, p4 = 100.000 38.115 156.875 123.027

Global Excess Supplies = es1, es2, es3, es4 = -0.00005 0.00008 -0.00002 0.00004

Production levels =

x11, x12, x13, x14 = 1.591 2.425 0.000 0.000
x21, x22, x23, x24 = 0.000 1.748 1.799 0.000
x31, x32, x33, x34 = 0.000 0.000 1.075 3.373

Excess supply levels =

s11, s12, s13, s14 = 0.872 0.540 -0.458 -0.292
s21, s22, s23, s24 = -0.872 -0.540 1.243 -0.709
s31, s32, s33, s34 = 0.000 0.000 -0.785 1.001

Consumption levels =

c11, c12, c13, c14 = 0.719 1.885 0.458 0.292
c21, c22, c22, c24 = 0.872 2.288 0.556 0.709
c31, c32, c33, c34 = 0.000 0.000 1.860 2.372

GNP levels = w1, w2 ,w3 = 287.411 348.831 583.619

Utility levels = u1, u2, u3 = 0.652 0.886 2.100

Global GNP = 1219.861
Global Utility = 3.639

Case 1 (x22max and x23max have been changed)

Tariff (%) = 150

Production possibility frontiers parameters

----

x11max, x12max, x13max, x14max =	2.000	4.000	0.000	0.000
x21max, x22max, x23max, x24max =	0.000	4.000	2.000	0.000
x31max, x32max, x33max, x34max =	0.000	0.000	2.000	4.000

Total sum of squared excess supplies = tesopt = 4.259192187127558D-08

Equilibrium prices = p1, p2, p3, p4 = 100.000 37.959 153.057 118.086

Global Excess Supplies = es1, es2, es3, es4 = 0.00015 -0.00002 0.00001 -0.00014

Production levels =

x11, x12, x13, x14 =	1.593	2.419	0.000	0.000
x21, x22, x23, x24 =	0.000	1.777	1.792	0.000
x31, x32, x33, x34 =	0.000	0.000	1.088	3.357

Excess supply levels =

s11, s12, s13, s14 =	0.854	0.473	-0.483	-0.250
s21, s22, s23, s24 =	-0.854	-0.473	1.234	-0.723
s31, s32, s33, s34 =	0.000	0.000	-0.751	0.973

Consumption levels =

c11, c12, c13, c14 =	0.739	1.946	0.483	0.250
c21, c22, c22, c24 =	0.854	2.250	0.558	0.723
c31, c32, c33, c34 =	0.000	0.000	1.839	2.383

GNP levels = w1, w2 ,w3 = 295.419 341.701 562.862

Utility levels = u1, u2, u3 = 0.645 0.880 2.093

Global GNP = 1199.982

Global Utility = 3.618

Case 1 (x22max and x23max have been changed)

Tariff (%) = 200

Production possibility frontiers parameters

----

x11max, x12max, x13max, x14max =	2.000	4.000	0.000	0.000
x21max, x22max, x23max, x24max =	0.000	4.000	2.000	0.000
x31max, x32max, x33max, x34max =	0.000	0.000	2.000	4.000

Total sum of squared excess supplies = tesopt = 2.059154126645428D-08

Equilibrium prices = p1, p2, p3, p4 = 100.000 37.852 150.420 114.668

Global Excess Supplies = es1, es2, es3, es4 = 0.00008 -0.00001 0.00003 -0.00011

Production levels =

x11, x12, x13, x14 =	1.595	2.414	0.000	0.000
x21, x22, x23, x24 =	0.000	1.798	1.787	0.000
x31, x32, x33, x34 =	0.000	0.000	1.097	3.345

Excess supply levels =

s11, s12, s13, s14 =	0.842	0.426	-0.500	-0.219
s21, s22, s23, s24 =	-0.842	-0.426	1.227	-0.734
s31, s32, s33, s34 =	0.000	0.000	-0.726	0.953

Consumption levels =

c11, c12, c13, c14 =	0.753	1.988	0.500	0.219
c21, c22, c22, c24 =	0.842	2.224	0.560	0.734
c31, c32, c33, c34 =	0.000	0.000	1.823	2.392

GNP levels = w1, w2 ,w3 = 301.016 336.791 548.529

Utility levels = u1, u2, u3 = 0.636 0.875 2.088

Global GNP = 1186.336

Global Utility = 3.600

Consider optimization of utility in N1:

The optimal production decisions in N1,
the optimal consumption levels in N1,
the optimal import levels to N1,
and optimal export levels from N1,
are *all* functions of all parameters *in all* countries.

For instance, the analysis shows, that the optimal tariff, from the N1 perspective, on imports of product x4 from N3, is a function of the production possibility frontier *also* in N2.

One of many Conclusions:

It would be irrational to determine tariffs only based on parameters from the importing country and from the exporting country.

Abstract part 6

Linear and nonlinear optimization of international trade and tariffs, WSTA 2025, Peter Lohmander

- **The free trade equilibrium and the equilibrium affected by tariffs introduced by N1 are determined. The optimal tariff from the N1 perspective is determined. In free trade, N3 exports product 4 to N1 but N1 does not export anything to N3. Then, N1 optimizes a tariff on product 4, only imported from N3.**

Abstract part 7

Linear and nonlinear optimization of international trade and tariffs, WSTA 2025, Peter Lohmander

- **The optimal tariff from the N1 perspective is strictly positive and the utility of N1 increases compared to the free trade solution. The total utility of all consumers in all countries decreases when the tariff increases. The optimal tariff determined by N1 is much larger than in the linear model, and is a function of all parameters in all countries. Several other tariff effects on all countries and products, including production and consumption changes in N2, are reported.**

Thank you very much for your time, questions and a Great Conference!