Optimal Strategies and Tactical Decisions for Army Brigades

Peter Lohmander

Prof Dr Peter Lohmander
Peter Lohmander Optimal Solutions
Sweden

$$\begin{cases} \dfrac{dx}{dt} = u - \alpha(Depth, FWA, FWP)\, y \\ \dfrac{dy}{dt} = v - \beta(Depth, FWA, FWP)\, x \end{cases}$$

Version 260623

Summary

This presentation shows new methods for optimizing delay operations with brigades.

The objective function takes into account all the costs of the war and its consequences.

Part 1: In advance, at the strategic level: Determination of the optimal depth of defense, the optimal initial number of soldiers, the optimal investments in fieldworks for defense, as well as fieldworks for attacks.

Part 2: Gradually, at the tactical level: Determination of how the different battalions should allocate their total capacity, i.e., the use of time and thus also other resources.

The optimal strategic and tactical decisions are influenced by the prices of various resources and measures. Examples of how different strategic and tactical decisions are affected by various conditions are presented.

References https://www.lohmander.com/Information/Ref.htm

Optimal Strategies and Tactical Decisions for Army Brigades

Peter Lohmander

Optimal Solutions, Hoppets Grand 6, SE 903 34 Umea, Sweden

Abstract

Strategies and tactical decisions, of an army brigade, are optimized. The objective function is based on all costs of the preparations and of the war. In the first section, the optimal depth of defense, Depth, is determined. Simultaneously, the optimal levels of field work for attack, FWA, and for protection, FWP, are derived. FWA includes defense preparations with vehicle mines, antipersonnel mines, destruction of bridges and preparations of assaults and counter attacks. Construction of fortifications and tunnels are parts of FWP. The optimal initial number of defending soldiers, x_0, is simultaneously calculated. A differential equation system model of an extended Lanchester type is constructed and solved, with attrition parameters that are functions of Depth, FWA and FWP. The differential equation system model is used as a sub routine in a new defense optimization model. The time before the war ends, T, and the number of unharmed defending soldiers, x_T, are simultaneously determined as functions of the parameters and optimal decisions. The optimal objective function value is a decreasing function of the price of the land and of the price of the soldiers. In optimum: Depth and x_0 are decreasing functions of the respective prices; FWP is an increasing function of the prices of land and soldiers, and FWA is an increasing function of the price of land. In the second section, the activities within the defending brigade, BLUE, and the attacking force, RED, are analyzed in detail. Artillery battalions, mechanized battalions and cavalry ranger battalions cooperate with each other and with other kinds of units. A controlled discrete time nonlinear difference equation system in six dimensions is constructed and numerically solved. This model system represents the war that takes place within the area defended by the BLUE brigade. The model describes the dynamic developments of the BLUE and RED resources. The model optimizes the degree of cooperation and selection of targets by different kinds of battalions. The attrition levels are nonlinear functions of the amounts of time that different kinds of battalions spend in different ways in this cooperation. For instance, a cavalry ranger battalion may search and find targets for an artillery battalion, and/or attack enemy cavalry units or other enemies. Each battalion has limited total time available. The model is used to derive the dynamic development of the resources of BLUE and RED under different parameter assumptions, representing dynamic unit deployments and other conditions. The model can with marginal adjustments also be used to determine the dynamic game outcomes when BLUE and RED sequentially adapt the decisions to the changes made by the other sides. As an illustration, the optimal use of cavalry ranger battalion time as a function of the cost per unit of the remaining enemy artillery battalions at a future point in time, is determined.

Keywords

Military optimization, strategy, tactics, coordination, army, brigade, battalion.

CONTACT Peter Lohmander. Email: Peter@Lohmander.com

Connected articles:

Lohmander, P. (2017). Applications and Mathematical Modeling in Operations Research, In: Cao BY. (ed) Fuzzy Information and Engineering and Decision. IWDS 2016. ***Advances in Intelligent Systems and Computing***, vol 646. Springer, Cham, 2018 Print ISBN 978-3-319 66513-9, Online ISBN 978-3-319-66514-6, Pages 46-53, eBook Package: Engineering, https://doi.org/10.1007/978-3-319-66514-6_5

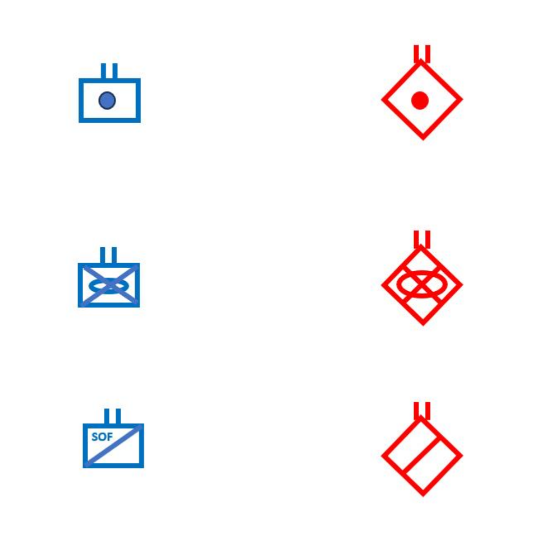
Lohmander, P. (2019a). Optimal decisions and expected values in two player zero sum games with diagonal game matrixes, - Explicit functions, general proofs and effects of parameter estimation errors, ***International Robotics & Automation Journal***, Volume 5, Issue 5, 2019, pages 186-198. https://medcraveonline.com/IRATJ/IRATJ-05-00193.pdf

Lohmander, P. (2019b). Four central military decision problems, General methods and solutions, ***The Royal Swedish Academy of War Sciences Proceedings and Journal***, 2/2019, pages 119-134. https://kkrva.se/hot/2019:2/09-lohmander-fyra-centrala-militara-beslutsproblem.pdf

Lohmander, P. (2023). Optimal Dynamic Control of Proxy War Arms Support, ***Automation***, 2023, 4, pages 31-56. Publisher: MDPI Journals, Zürich, Switzerland. https://doi.org/10.3390/automation4010004 https://www.mdpi.com/2673-4052/4/1/4

Lohmander, P. (2024). Attrition coefficient estimations via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions, ***Journal of Statistics and Computer Science***, Vol. 3, Issue 1, 2024, pages 51-78. Publisher: ARF India. https://www.arfjournals.com/image/catalog/Journals%20Papers/JSCS/2024/No%201%20(2024)/ART_4.pdf

Lohmander, P. (2025). Optimal Deployment, ***Mathematics and Systems Science***, 2025, 3(1), 3055, pages 1–46. Publisher: Asia Pacific Academy of Science Pte. Ltd., Singapore. http://doi.org/10.54517/mss3055

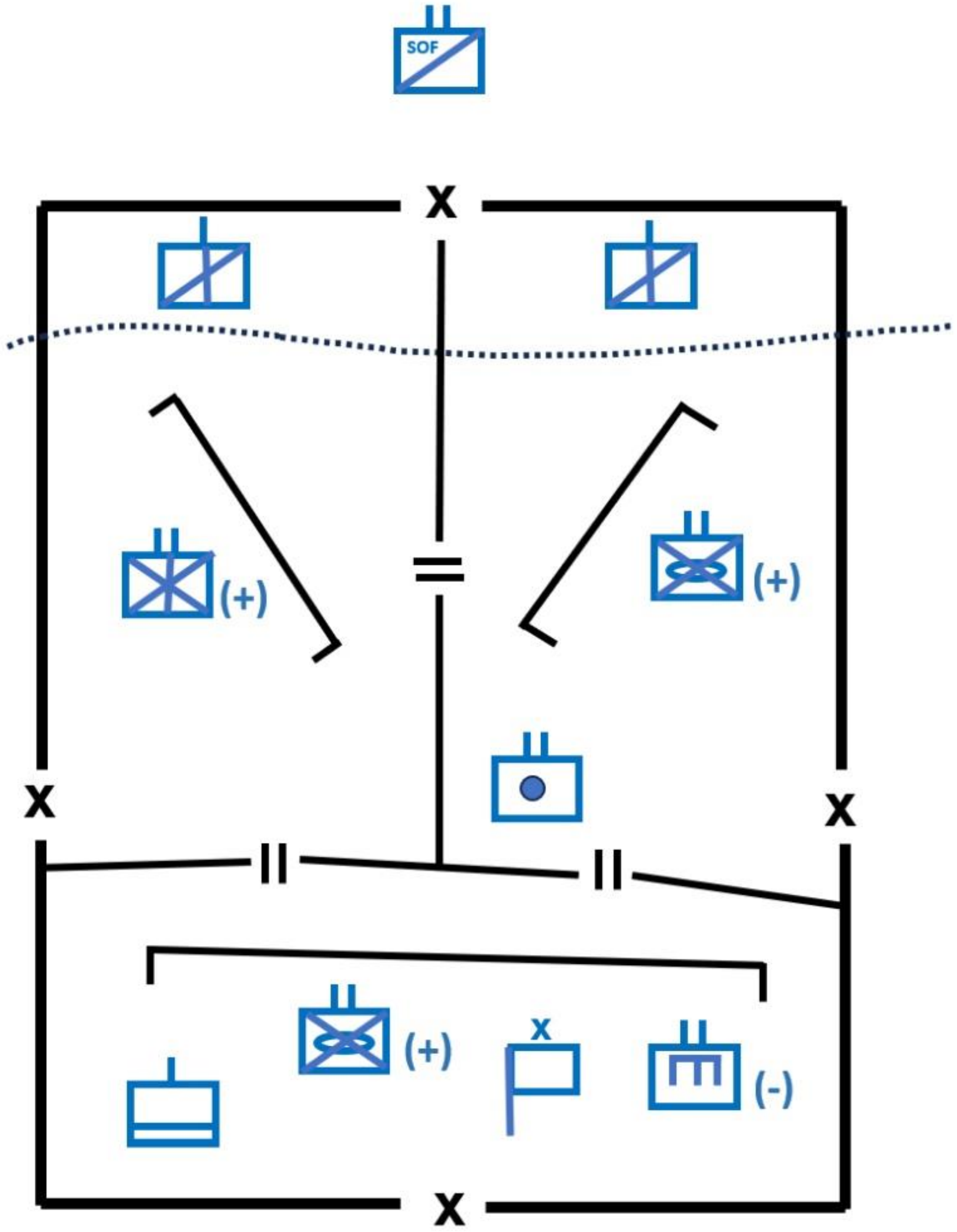


Artillery battalion

Mechanized battalion

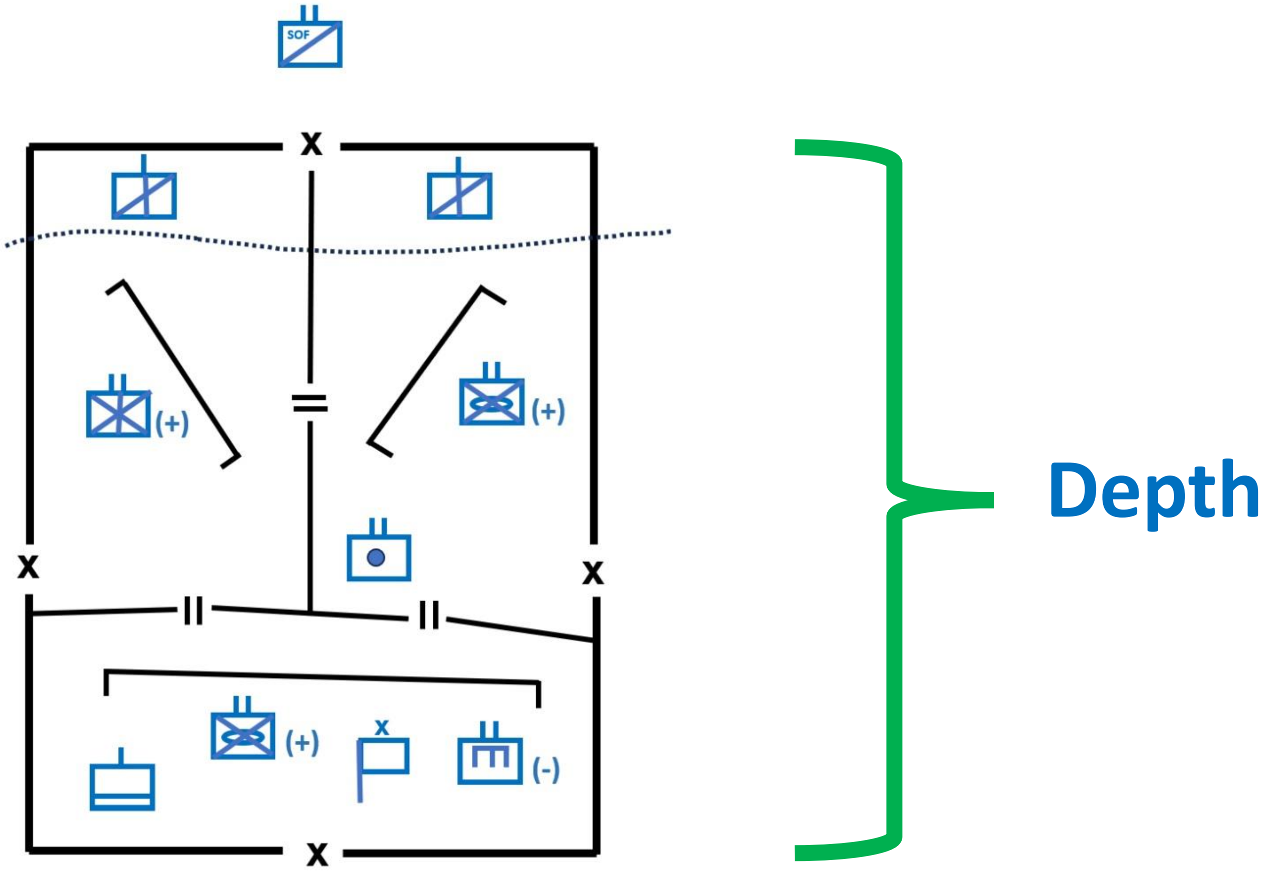
Ranger battalion

A Brigade Defense Area



Part 1:

Optimization of the depth of defense, Depth.



Simultaneously, the optimal levels of field work for attack, FWA, and protection, FWP, are determined.

FWA

FWP

Simultaneously, the optimal initial number of defending soldiers, x0, is also determined.

x0 →

A differential equation system model of extended Lanchester type is constructed and solved,

with attrition coefficients that are *functions* of Depth, FWA and FWP.

The differential equation system model is used as a sub routine in a new defense optimization model.

The Classical Lanchester Model

Time derivative of the size of the defending force

$$\begin{cases} \dfrac{dx}{dt} = -\alpha y \\[2ex] \dfrac{dy}{dt} = -\beta x \end{cases}$$

Size of the attacking force

Attrition coefficients

Time derivative of the size of the attacking force

Size of the defending force

The Extended Lanchester Model

Reinforcements per time unit of the defending force

Time derivative of the size of the defending force

$$\begin{cases} \dfrac{dx}{dt} = u - \alpha\, y \\[2ex] \dfrac{dy}{dt} = v - \beta\, x \end{cases}$$

Size of the attacking force

Attrition coefficients

Time derivative of the size of the attacking force

Size of the defending force

Reinforcements per time unit of the attacking force

The Extended Lanchester Model with attrition coefficients that are functions of decision variables.

$$\begin{cases} \dfrac{dx}{dt} = u - \overbrace{\alpha(Depth, FWA, FWP)}\,y \\ \dfrac{dy}{dt} = v - \beta(Depth, FWA, FWP)\,x \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = u - \alpha y & (1a) \\ \dfrac{dy}{dt} = v - \beta x & (1b) \end{cases}$$

The Extended Lanchester Model

The time path of (x(t), y(t))

$$\begin{cases} x(t) = A_1 e^{\lambda t} + A_2 e^{-\lambda t} + \dfrac{v}{\beta} \\ y(t) = -\dfrac{\lambda}{\alpha} A_1 e^{\lambda t} + \dfrac{\lambda}{\alpha} A_2 e^{-\lambda t} + \dfrac{u}{\alpha} \end{cases}$$

$$\lambda = \sqrt{\alpha\beta}$$

With the initial conditions, $(x(0),y(0)) = (x_0, y_0)$, we may determine (A_1, A_2) from this linear 2x2 equation system.

$$\begin{bmatrix} e^{\lambda t} & e^{-\lambda t} \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{\lambda t} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} e^{-\lambda t} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} x(t) - \frac{v}{\beta} \\ y(t) - \frac{u}{\alpha} \end{bmatrix}$$

$$A_1 = \frac{\begin{vmatrix} x(0) - \frac{v}{\beta} & 1 \\ y(0) - \frac{u}{\alpha} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} \end{vmatrix}} = \frac{\left(x(0) - \frac{v}{\beta}\right)\frac{\sqrt{\beta}}{\sqrt{\alpha}} - y(0) + \frac{u}{\alpha}}{2\frac{\sqrt{\beta}}{\sqrt{\alpha}}}$$

$$A_2 = \frac{\begin{vmatrix} 1 & x(0) - \frac{v}{\beta} \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} & y(0) - \frac{u}{\alpha} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -\frac{\sqrt{\beta}}{\sqrt{\alpha}} & \frac{\sqrt{\beta}}{\sqrt{\alpha}} \end{vmatrix}} = \frac{y(0) - \frac{u}{\alpha} + \frac{\sqrt{\beta}}{\sqrt{\alpha}}\left(x(0) - \frac{v}{\beta}\right)}{2\frac{\sqrt{\beta}}{\sqrt{\alpha}}}$$

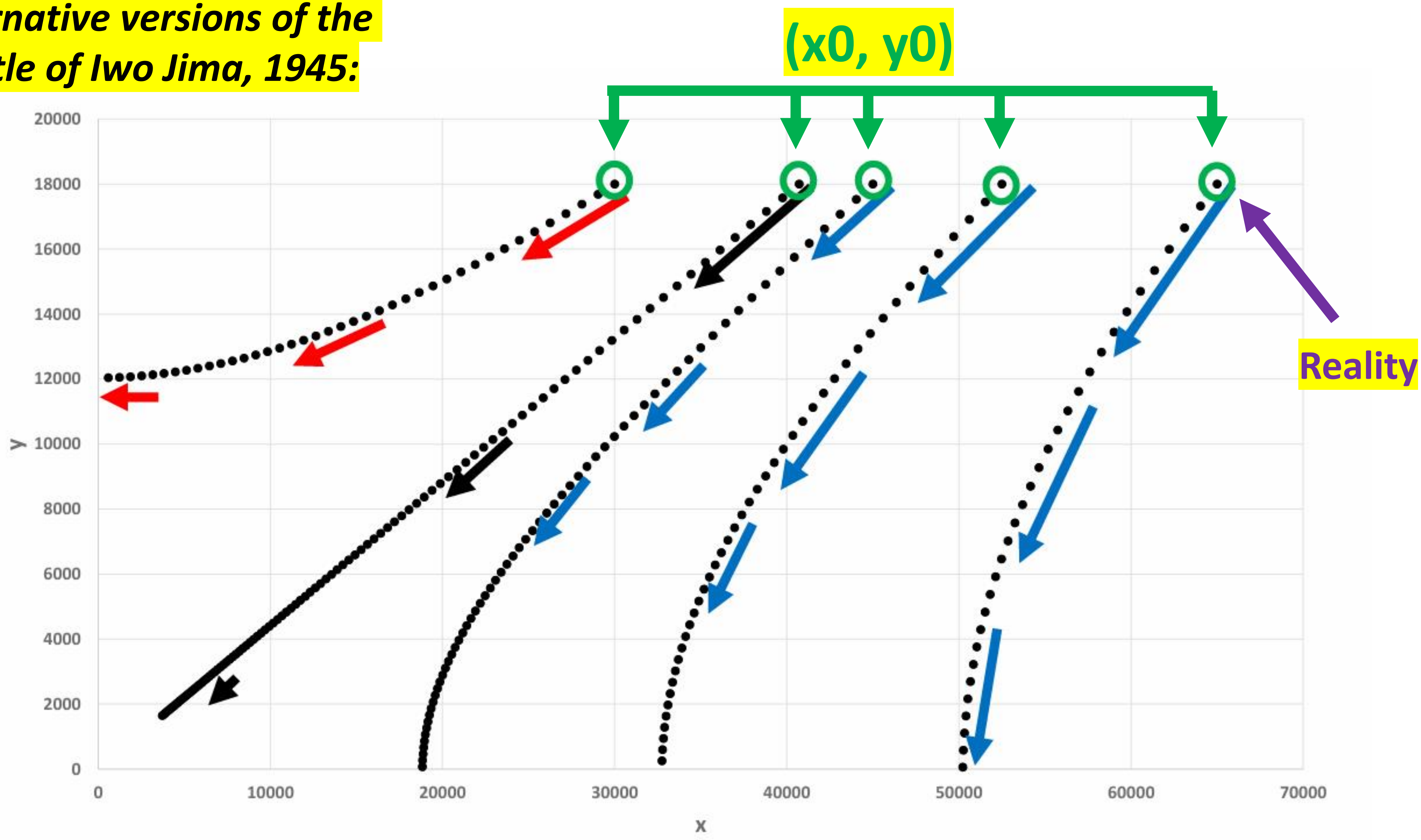
x(t) and **y(t)** are functions of A_1, A_2 and more.

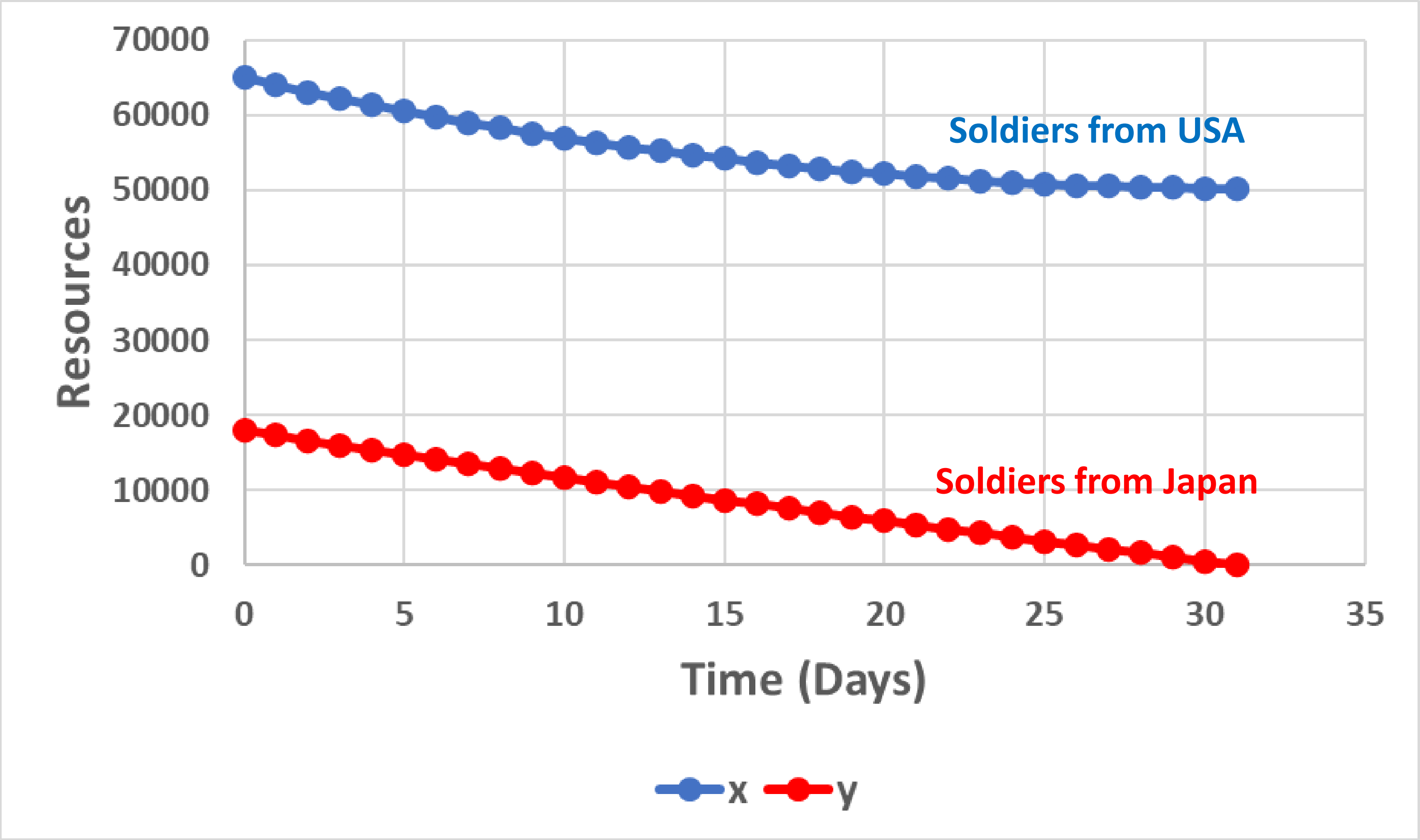
The parameters A_1 and A_2 are functions of the parameters of the Extended Lanchester differential equation system and of the initial conditions (x(0),y(0)) = (X_0,Y_0)

Furthermore, the parameters of the Extended Lanchester differential equation system are functions of several decision variables.

(Depth, FWA, FWP)

Alternative versions of the Battle of Iwo Jima, 1945:





Numbers from the Battle of Iwo Jima, 1945:

0.05347

$$\begin{cases} \dot{x} = -ay \\ \dot{y} = -bx \end{cases}$$

0.01045

Source:

Lohmander, P., Attrition coefficient estimations via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions,
Journal of Statistics and Computer Science, Vol. 3, Issue 1, 2024, pp. 51-78.
https://www.arfjournals.com/jscs/issue/322
https://www.arfjournals.com/image/catalog/Journals%20Papers/JSCS/2024/No%201%20(2024)/ART_4.pdf

The attrition coefficients are functions of Depth, FWP and FWA.

(Functions of these types can and should be determined with local empirical data.)

alpha = alpha_a * (50 / Depth) * (1 - .3 * FWP ^ .5)

constant — *decreases with Depth* — *decreases with FWP*

constant — *increases with Depth* — *increases with FWA*

beta = beta_a * (Depth / 50) * (1 + .3 * FWA ^ .5)

Master Decision Optimization Problem:

$$\max_{Depth, FWA, FWP, x_0} \phi(\bullet; P_{Depth}, P_{FWA}, P_{FWP}, P_{x_0}, \ldots)$$

Sub Routine:

All results of the conflict are calculated, based on the Extended Lanchester differential equation system, all parameters, and decision variables.

Answers sent to the Master Decision Optimization Problem:

Will x(t) or y(y) reach size zero first? What is then the value of t, called T?

What will be the size of the force that does not reach zero?

What are the costs of different used and lost resources etc.

Example of Objective function in the Master Problem:

Objective =

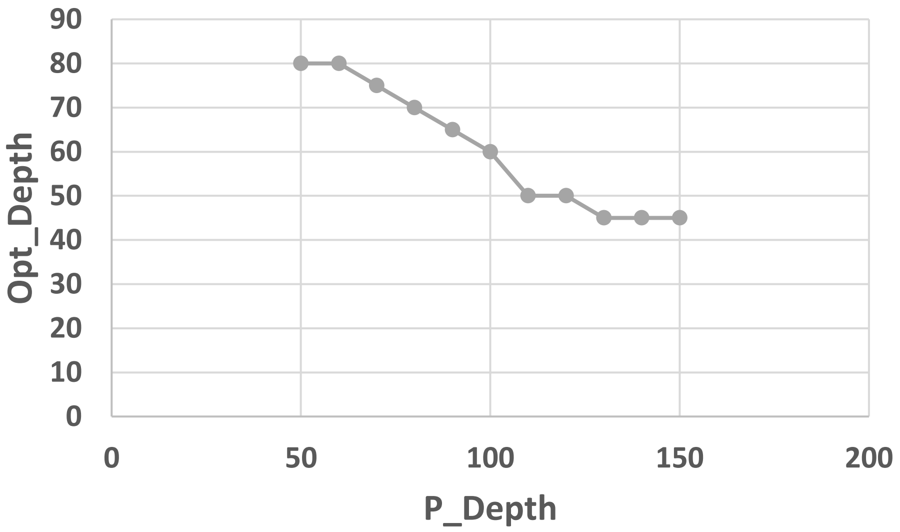
- P_Depth * Depth
- P_x0 * x0
- .0002 * x0 * x0
- 1 * u * T
- 170 * FWP - 170 * FWA
+ Exp(-r * T) * (xT - yT)
- 10000 * ywins

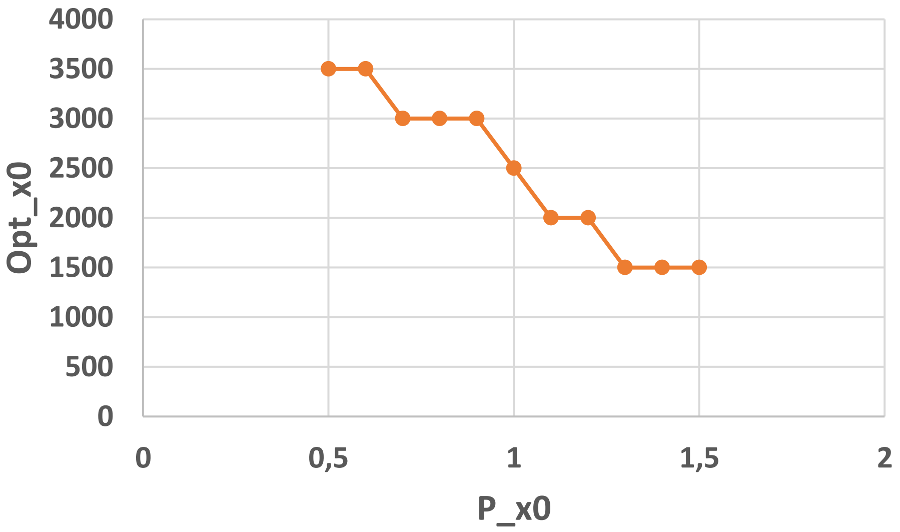
Decision variables that the defending nation can control.

Decision variables that a coalition can control.

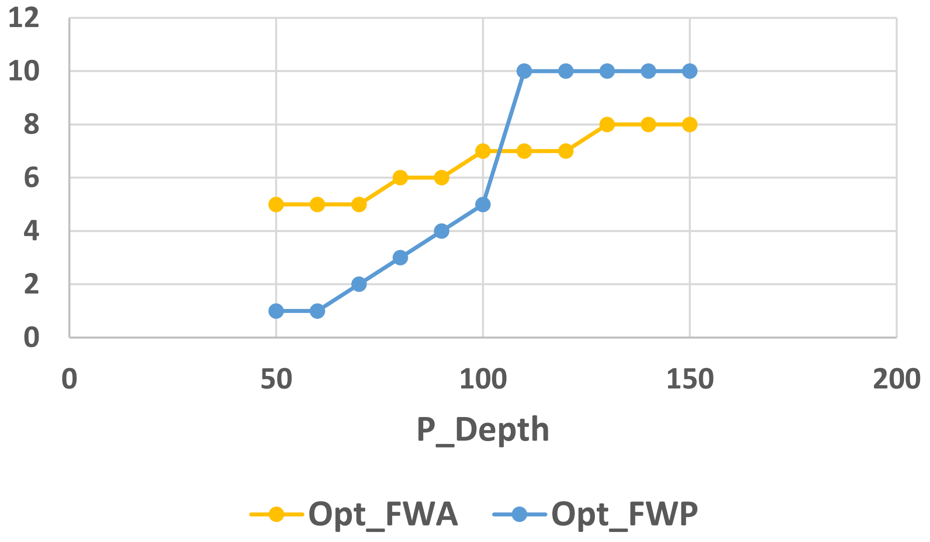
Optimal decisions:

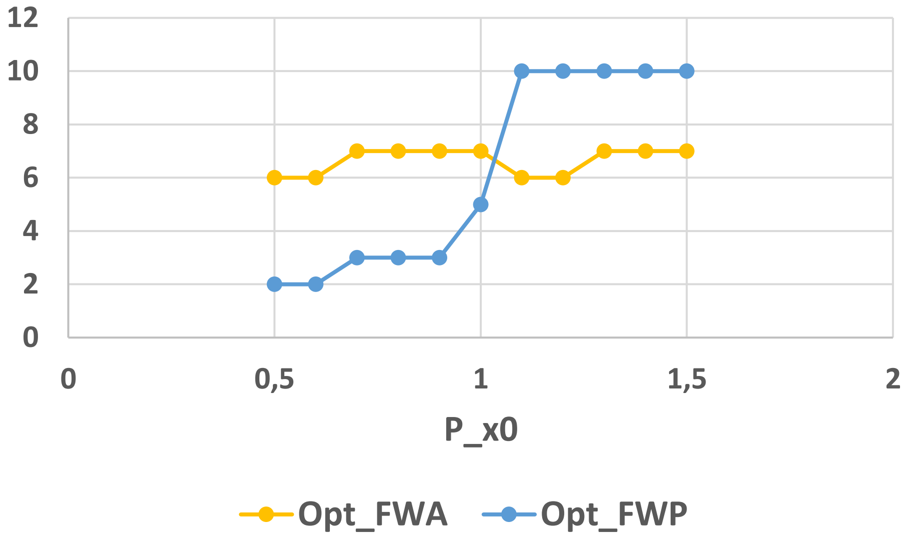
Opt_Depth and Opt_x0 are *decreasing functions* of the respective prices.





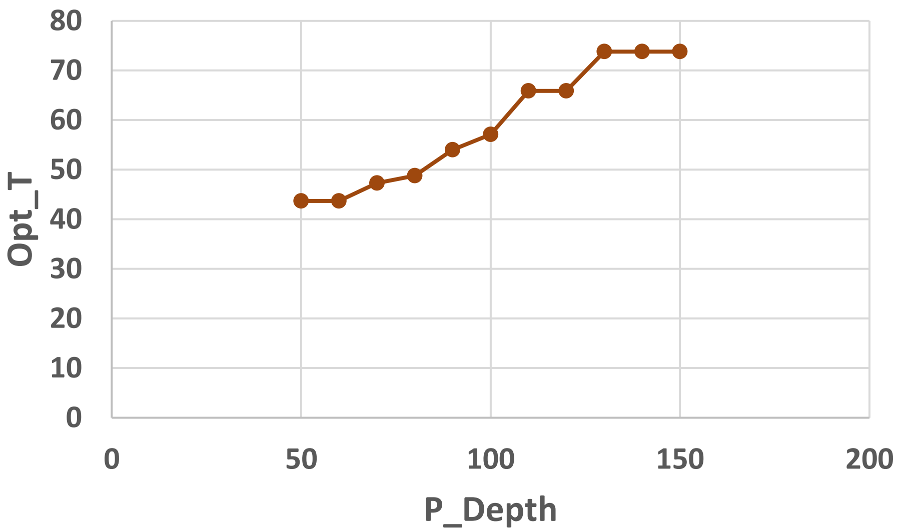
Opt_FWP is an ***increasing function*** of the prices for land and soldiers.

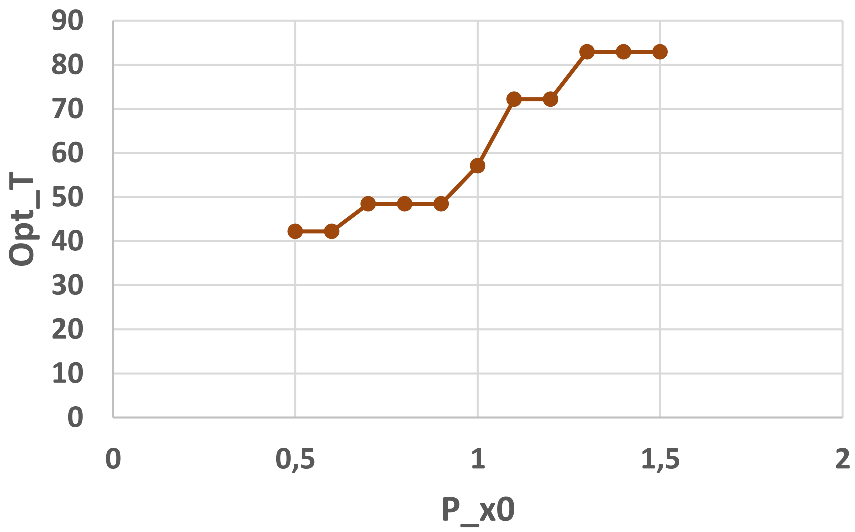




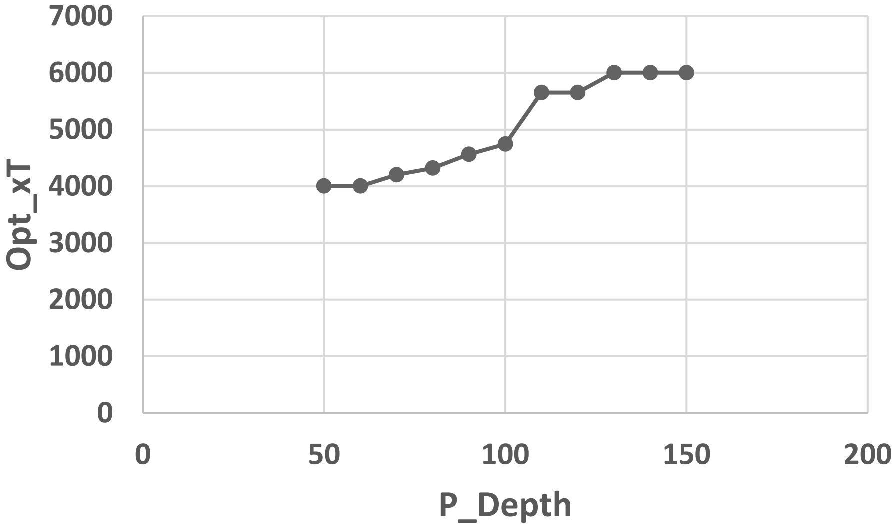
Opt_FWA är is an ***increasing function*** of the price of land.

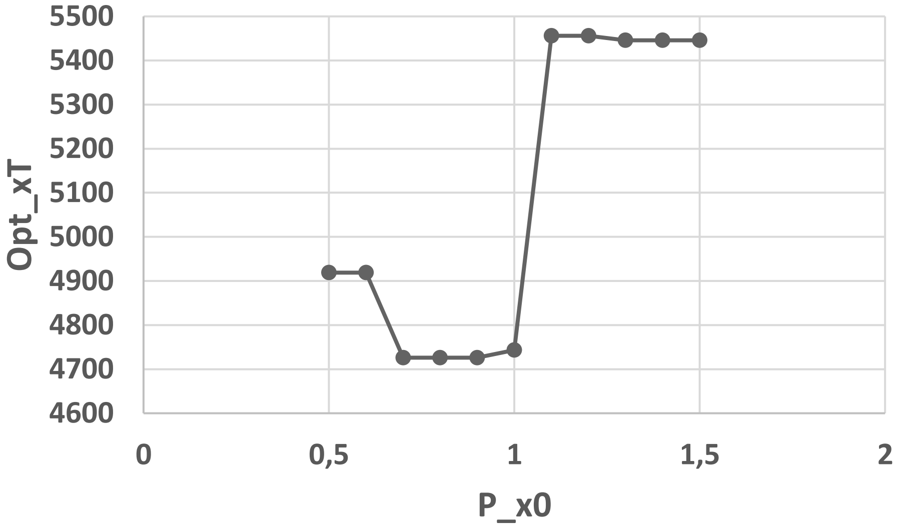
T, the time when the war ends, (when x or y is zero) is determined as a ***<u>function of the parameters and the optimal decisions</u>***.



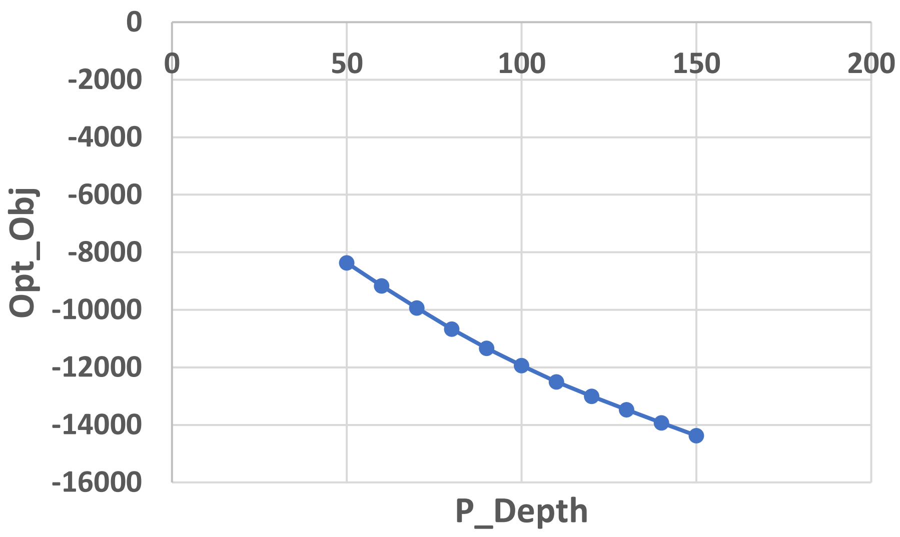


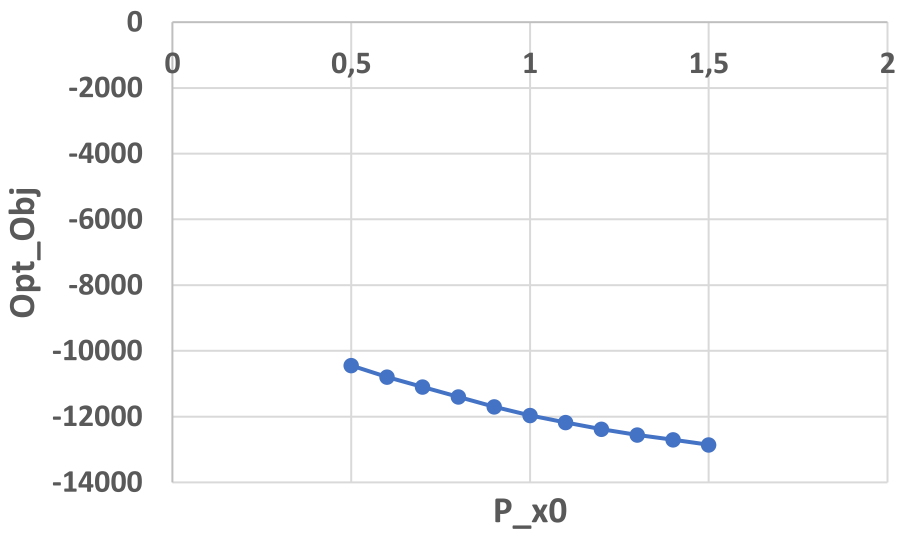
The number of unharmed defending soldiers, when the war ends, xT, is a *<u>function of the parameters and the optimal decisions.</u>*



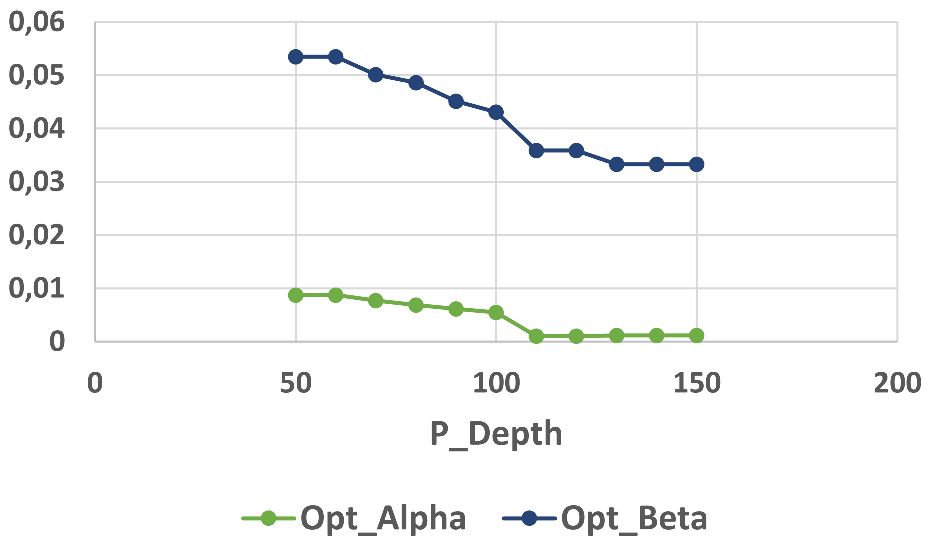


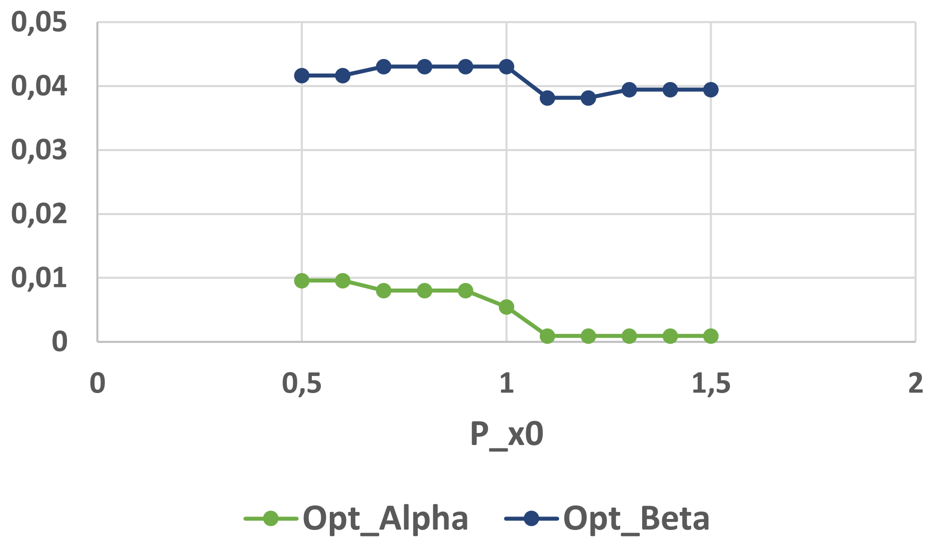
The optimal objective function value, Opt_Obj, is a decreasing function of the prices of land and soldiers.





The optimal value of the attriction coefficient that shows how rapidly the size of the defending force is reduced, Alpha, is a decreasing function of the prices of land and soldiers.





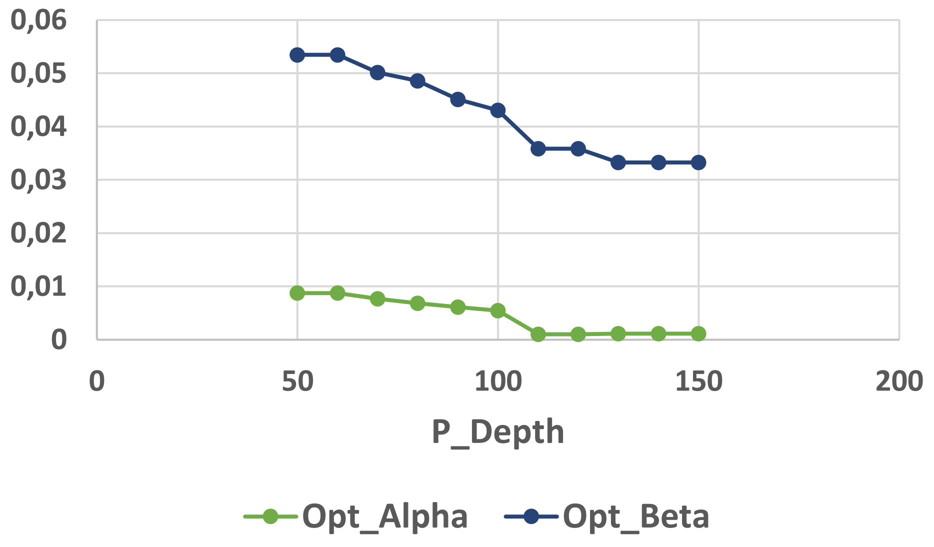
The optimal value of the attrition coefficient that shows how rapidly the size of the attacking force is reduced, Beta, is a decresing function of the price of land.

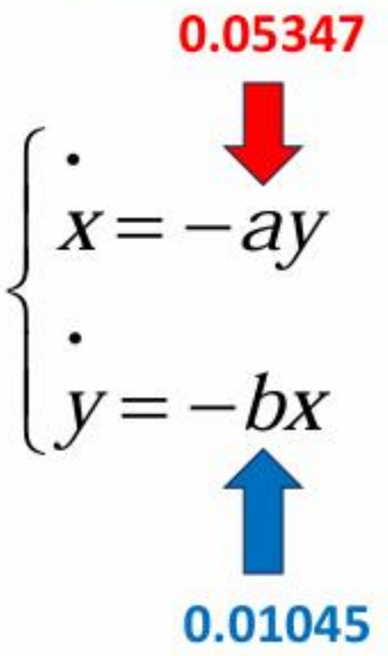
Observations:

Lohmander, P. (2024). Attrition coefficient estimations via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions, ***Journal of Statistics and Computer Science***, Vol. 3, Issue 1, 2024, pages 51-78. Publisher: ARF India. https://www.arfjournals.com/image/catalog/Journals%20Papers/JSCS/2024/No%201%20(2024)/ART_4.pdf

Left Graph:
Assume that P_Depth = 50.
(P_x0 = 1, which is not seen the the graph).
Alpha = 0.88% and Beta = 5.35%

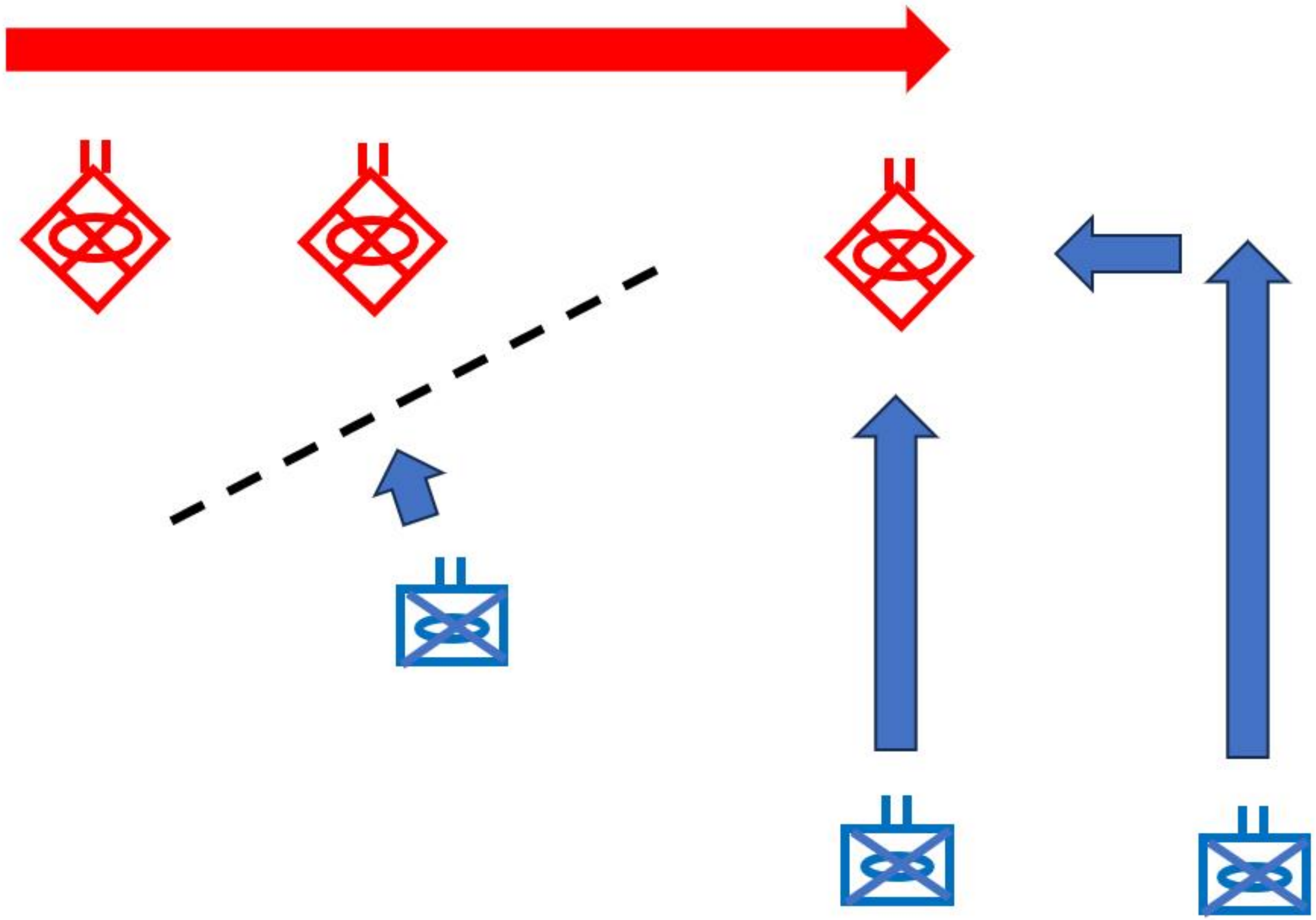
Then, a defending soldier deletes 6.08 attacking soldiers when one attacking soldier deletes one defending soldier. Compare the attack by USA on Iwo Jima during WW II. See the differential equation system to the right. Then, the situation was almost the same.



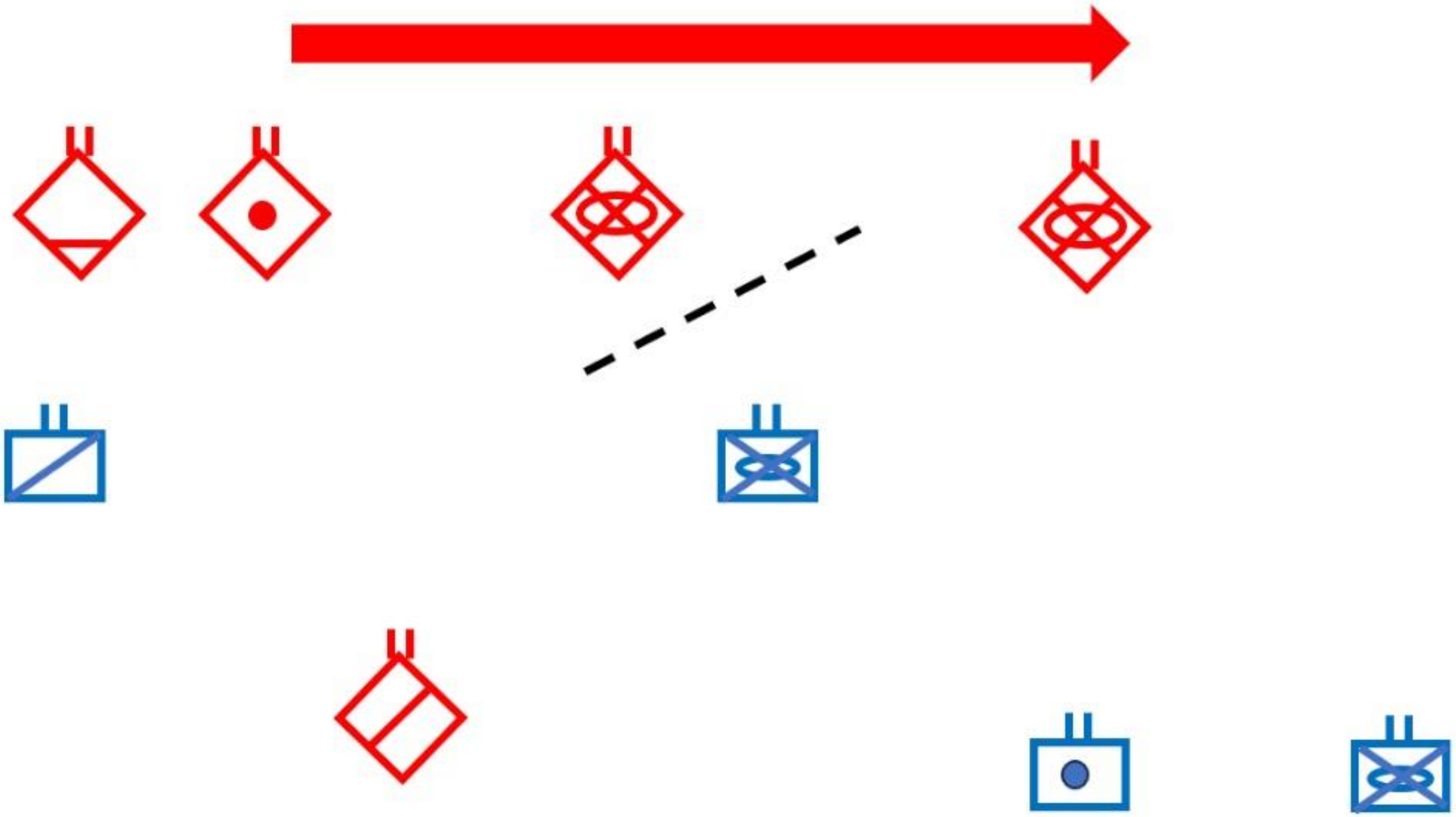


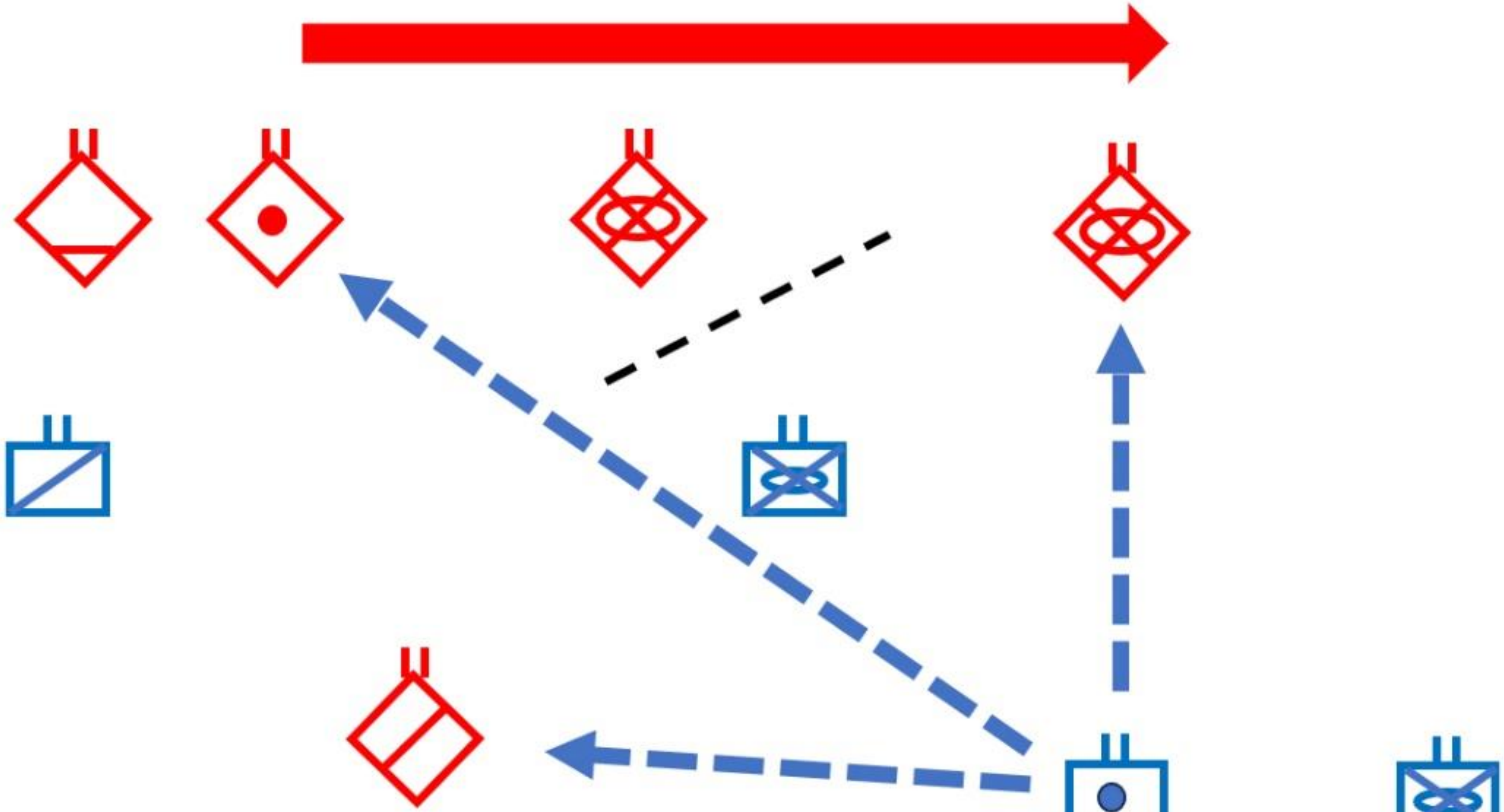
$$\begin{cases} \dot{x} = -ay \\ \dot{y} = -bx \end{cases}$$

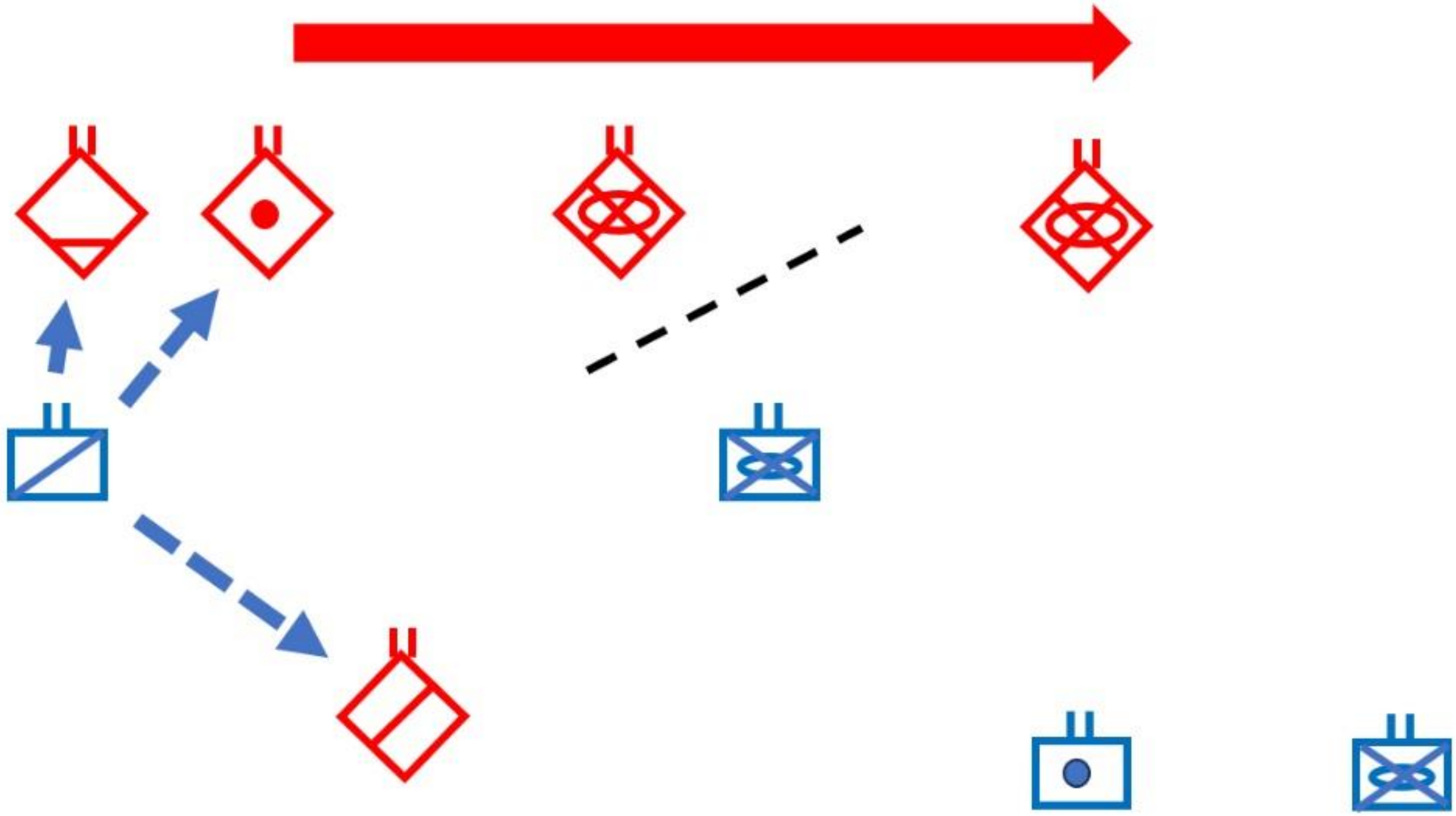
Brigades and dynamic cooperation between battalions

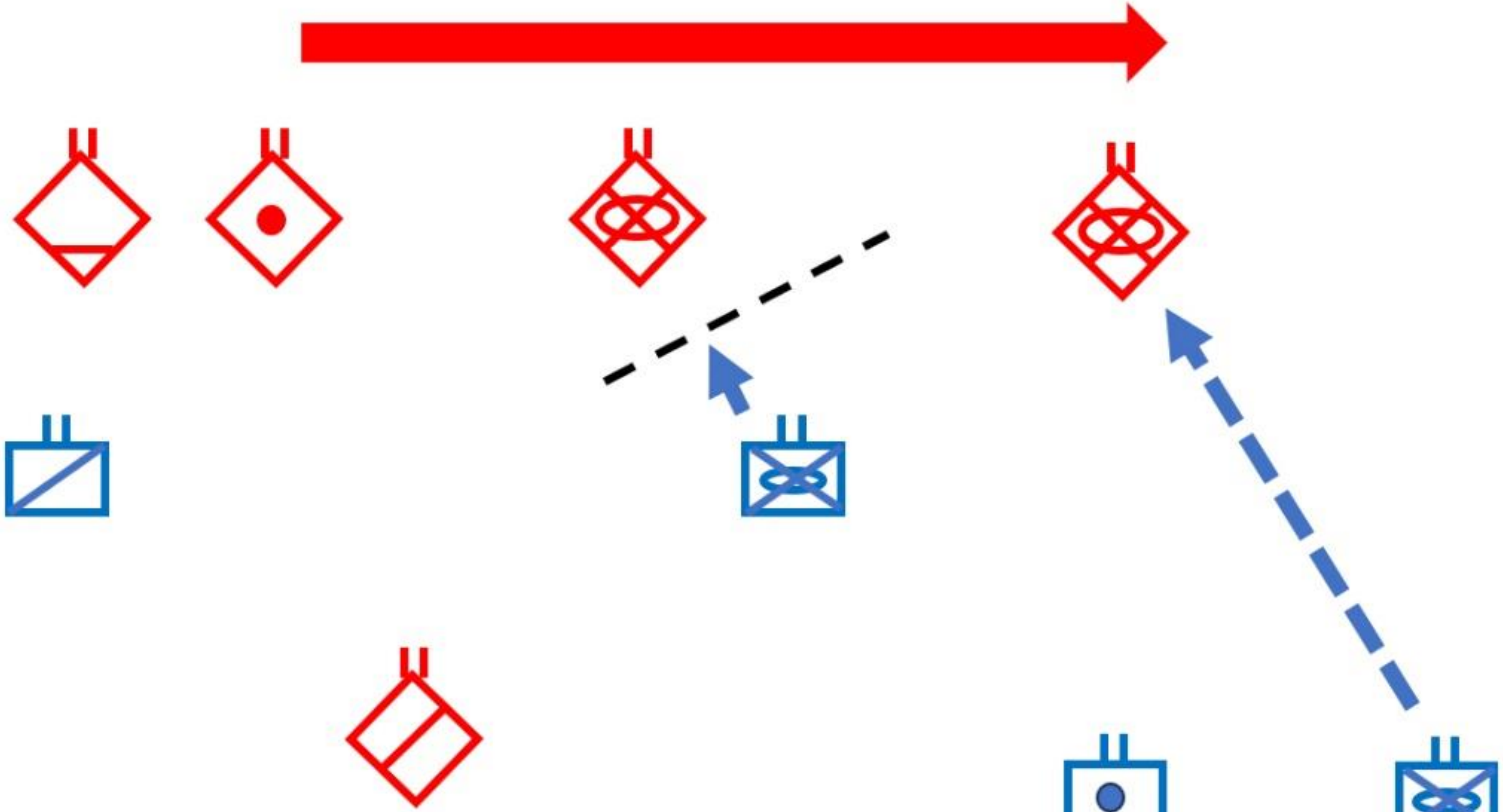


Alternative situation



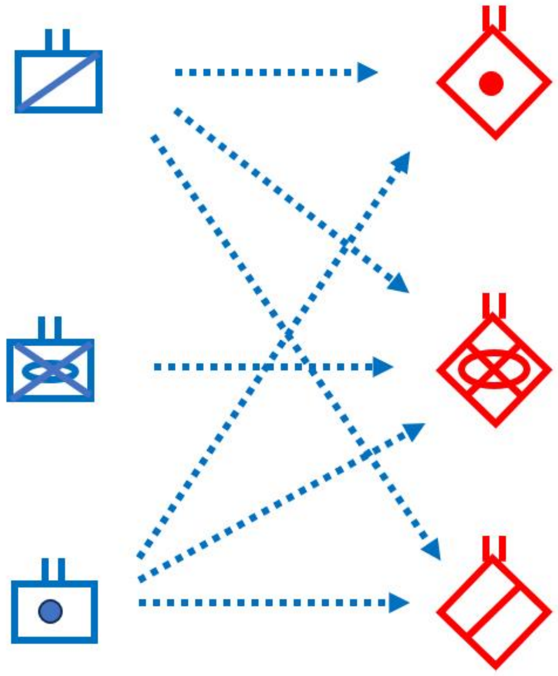


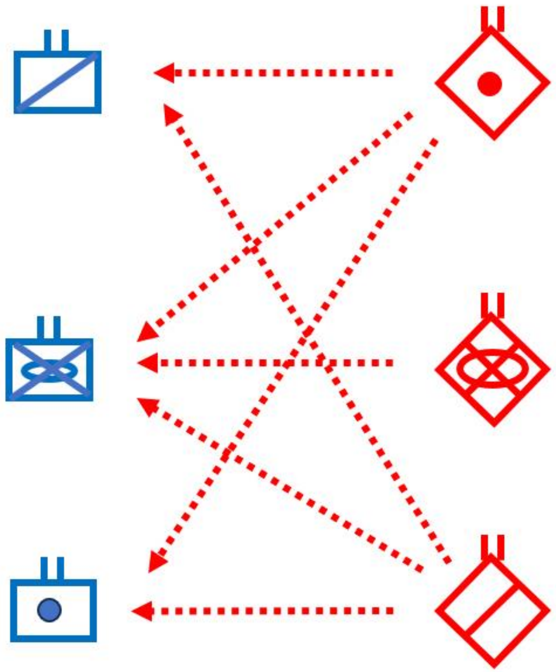




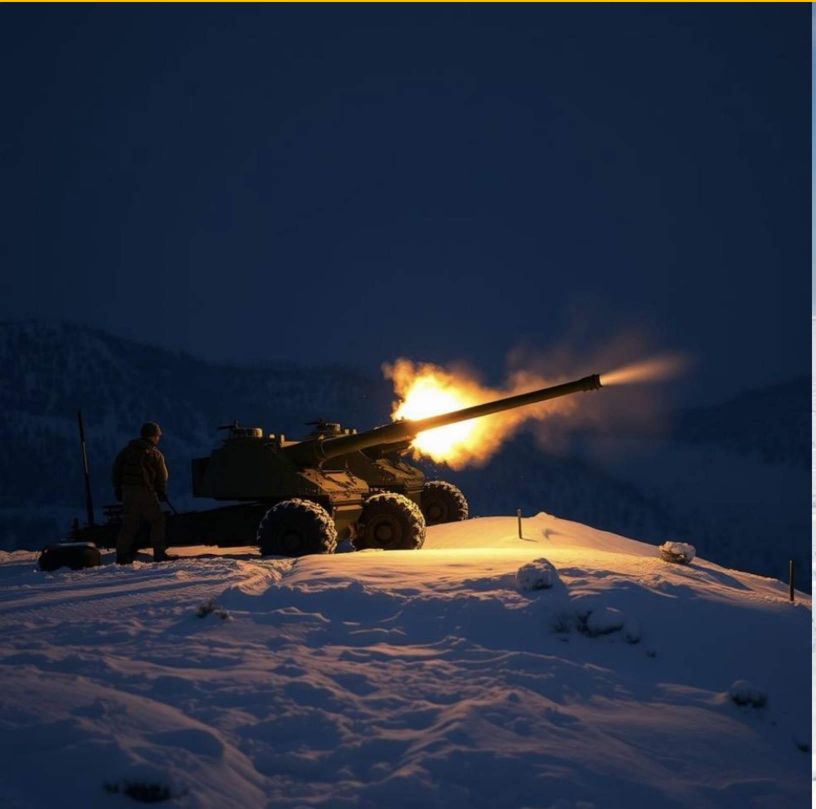
Part 2:

The acivitites within the defending Blue and the attacking Red Brigades are analyzed in detail.





Artillery battalions, mechanized battalions and ranger battalions cooperate with each other.



A controlled discrete time nonlinear difference equation system in six dimensions is constructed and numerically solved.

$$dX_A(t) = jX_A(t) - \phi_{X_A}(v_{AA}, v_{CA}, Y_A, Y_C) \quad (44)$$

$$dX_C(t) = jX_C(t) - \phi_{X_C}(v_{AC}, v_{CC}, Y_A, Y_C) \quad (45)$$

$$dX_M(t) = jX_M(t) - \phi_{X_M}(v_{AM}, v_{CM}, v_{MM} Y_A, Y_C, Y_M) \quad (46)$$

$$dY_A(t) = jY_A(t) - \phi_{Y_A}(u_{AA}, u_{CA}, X_A, X_C) \quad (47)$$

$$dY_C(t) = jY_C(t) - \phi_{Y_C}(u_{AC}, u_{CC}, X_A, X_C) \quad (48)$$

$$dY_M(t) = jY_M(t) - \phi_{Y_M}(u_{AM}, u_{CM}, u_{MM}, X_A, X_C, X_M) \quad (49)$$

This model represents the part of the war that occurs within the area defended by the Blue Brigade.

The model describes the dynamic developments of the Blue and Red resources.

The model optimizes the degree of cooperation and selection of targets for different types of battalions.

The attrition levels are nonlinear functions of the time that different types of battalions spend in different ways in this cooperation.

For instance, a Blue ranger battalion can search and find targets for a Blue artillery battalion and/or attack Red rangers or other Red units.

Every battalion has limited total time available.

For example: A Blue ranger battalion can search and find targets for a Blue artillery battalion.

A Blue ranger battalion can attack Red forces.

A Blue ranger battalion can destroy Red tanks.


A Blue ranger battalion can destroy a bridge.

A Blue ranger battalion can use drones to destroy Red fuel deliveries.

Possible extension:

With marginal adjustments, the model can be used to determine the outcomes and the optimal decisions if Blue and Red dynamically adjust the decisions.

Illustration:

The optimal use of time of a Blue Ranger battalion, as a function of the cost per unit of the remaining Red artillery battalions at a future point in time, T.

Examples of objective functions

Red Artillery battalions
Red Cavalry Rangers battalions
Red Mechanized battalions

$$W_1 = f_{Y_A} Y_A(T) + f_{Y_C} Y_C(T) + f_{Y_M} Y_M(T) \quad (60)$$

$$\left(f_{Y_A}, f_{Y_C}, f_{Y_M} \right) = \left(-10, -5, -10 \right) \quad (61)$$

This ”cost per unit” will be adjusted.

Optimal share of time of the Blue Ranger battalion, to fight Red Cavalry Rangers.

Table 1.

Cost (for Blue) per unit of Red Artillery battalion at time T.

```
Parameter fYa = -15

 Optimal Solution:

  objective function = -1085.318

  uca       ucc       ucm
  0.75      0.03      0.22

            Art        Cav        Mec
X(topt) =   9.557317   12.77866   40.5522
Y(topt) =   9.670883   29.32049   79.3652
```

Optimal share of time of the Blue Ranger battalion, to fight Red Mechanized units.

Optimal share of time of the Blue Ranger battalion, to search for targets for the Blue artillery battalion to fight Red artillery.

Estimated remaining number of Red artillery Battalions at time T.

Table 2.

Cost (for Blue) per unit of Red Artillery battalion at time T.

```
Parameter fYa = -10

  Optimal Solution:

   objective function = -1031.586

   uca      ucc      ucm
   0.60     0.06     0.34

            Art        Cav        Mec
X(topt) =  9.185617   12.59281   40.33058
Y(topt) =  12.17004   28.23752   76.86983
```

Optimal share of time of the Blue Ranger battalion, to search for targets for the Blue artillery battalion to fight Red artillery.

Estimated remaining number of Red artillery Battalions at time T.

Cost (for Blue) per unit of Red Artillery battalion at time T.

Table 3.

```
Parameter fYa = -5

  Optimal Solution:

   objective function = -959.2858

   uca      ucc      ucm
   0.32     0.11     0.57

            Art       Cav       Mec
  X(topt) = 8.382638  12.19132  39.83494
  Y(topt) = 17.75569  26.98911  73.55618
```

Optimal share of time of the Blue Ranger battalion, to search for targets for the Blue artillery battalion to fight Red artillery.

Estimated remaining number of Red artillery Battalions at time T.

Observations:

If it becomes more important for the Blue brigade to reduce the number of Red artillery battalions at time T = 100, (f_{Ya} goes from -5 to -15) ,

Then,

The Blue rangers should increase the relative time spent on searching for Red artillery locations, to be reported to the Blue artillery.

f_{Ya}	Optimal share of time	Red artillery Battalions at time T
-5	32%	18
-10	60%	12
-15	75%	10

Conclusions:

It is possible to optimize delay operations with army brigades.

All costs of the war and the consequences may be considered.

In advance, at the strategic level, we should determine the optimal depth of defense, the optimal initial number of soldiers, and optimal investments in field works for defense and attack

Over time, at the tactical level, we should optimize the use of time (capacity) of the different battalions.

The optimal strategic and tactical decisions are strongly affected by changes of prices for different kinds of resources and activities.

Examples of how optimal decisions are affected by prices and costs of different kinds have been given.

Connected presentations in Swedish:

Lohmander, P., Optimal Fördröjningsstrid med Brigad, Kamratforeningen Blå Dragoner, Umeå Skvadron, 260416,
http://www.Lohmander.com/PL_K4_Brigad_26.pdf
http://www.Lohmander.com/PL_K4_Brigad_26.pptx

Lohmander, P., Optimal Krigföring via Tullar, Kamratforeningen Blå Dragoner, Umeå Skvadron, 251127,
http://www.lohmander.com/PL_K4_Tullar_25.pdf
http://www.lohmander.com/PL_K4_Tullar_25.pptx

Lohmander, P., Optimal Utnötning, från Iwo Jima till Ukraina och Framtiden, Kamratforeningen Blå Dragoner, Umeå Skvadron, 250327,
http://www.lohmander.com/PL_K4_25.pdf
http://www.lohmander.com/PL_K4_25.pptx

Lohmander, P., Optimal Jägarstrid, Kamratföreningen Blå Dragoner, Umeå Skvadron, Garnisonsmässen, Umeå, Sverige, Torsdagen den 18 April, 2024.
http://www.Lohmander.com/PL Optimal Jagarstrid 240418.pdf
http://www.Lohmander.com/PL Optimal Jagarstrid 240418.pptx

References:
https://www.lohmander.com/Information/Ref.htm

Thank you for your time!

Questions?

Peter@Lohmander.com
peter.lohmander@icloud.com